0731 $2.00

NASA CONTRACTOR REPORT

NASA CR-32

N64-17989

CODE 1

NASA CR-32

80p.

FREQUENCY RESPONSE FUNCTIONS AND COHERENCE FUNCTIONS FOR MULTIPLE INPUT LINEAR SYSTEMS

by Loren D. Enochson

Prepared under Contract No. NAS-5-4590 by
MEASUREMENT ANALYSIS CORPORATION
Los Angeles, California

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1964

80p refs

FREQUENCY RESPONSE FUNCTIONS

AND COHERENCE FUNCTIONS

FOR MULTIPLE INPUT LINEAR SYSTEMS

By Loren D. Enochson

Prepared under Contract No. NAS-5-4590 by

MEASUREMENT ANALYSIS CORPORATION

Los Angeles, California

This report is reproduced photographically
from copy supplied by the contractor.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. 20230 -- Price $2.00

FREQUENCY RESPONSE FUNCTIONS AND COHERENCE FUNCTIONS FOR MULTIPLE INPUT LINEAR SYSTEMS

CONTENTS

CONTENTS (Continued)

FREQUENCY RESPONSE FUNCTIONS AND COHERENCE FUNCTIONS FOR MULTIPLE INPUT LINEAR SYSTEMS

1. INTRODUCTION

17989

The main object of the subsequent discussion is the consideration of multiple input linear system frequency response functions and associated applications of coherence functions. Single input-single output linear systems are first described and pertinent definitions and formulas presented. These initial sections serve as an introduction to the multiple input linear system material. The coherence function for the single input-single output case is defined, and the behavior of this function is found for situations where noise is occurring in either or both the input and output measuring devices. This presentation demonstrates that the coherence function is a useful parameter in determining confidence limits for measurements of the gain factor and the phase angle of a linear system frequency response function.

AUTHOR

Multiple input linear systems with a single output are then discussed. Important formulas are given for power spectral density functions, autocorrelation functions, frequency response functions, and coherence functions in both the situation where the inputs are correlated and the situation where the inputs are independent. The consideration of the multiple input linear system case requires application of multi-dimensional random process theory. Also, complex variable multi-variate statistical analysis becomes extremely helpful here in that several pertinent analogies may be drawn which lead to significant results.

A central problem is that of the application of the coherence function as a detector of other inputs not being considered when only a single input and a single output are being measured. The system under consideration is assumed to be a constant parameter linear system with no noise occurring in the input or output measurements. It is shown that the ordinary coherence function will take on values less than unity when other inputs are occurring which are not being considered. Full knowledge of these other inputs for both correlated and independent cases requires calculation of more complicated partial coherence functions. The concept of partial coherence functions is explained here from basic principles.

Finally, a numerical engineering example is worked through to illustrate the various concepts and formulas developed in the theory. In the numerical example, both the case of correlated inputs and independent inputs are computed for the two input-single output linear system. It becomes immediately apparent that even in the simplest cases, the amount of computation involved for this type of problem is considerable so that a digital computer is an invaluable tool for these problems.

2. PROPERTIES OF FREQUENCY RESPONSE FUNCTIONS

Consider a physically realizable linear system which does not have any time varying parameters. The <u>weighting function</u> $h(\tau)$ associated with this system is defined as the response (i.e., the output function) of the system to a unit impulse input function and is measured as a function of time, τ, from the moment of occurrence of the impulse input. For physically realizable systems, it is necessary that $h(\tau) = 0$ for $\tau < 0$ since the response must follow the input. The usefulness of the concept of the weighting function is due to the following fact: the linear system is completely characterized by its weighting function in the sense that given any arbitrary input $x(t)$ known as a function of time t for all t, the system output $y(t)$ is determined by the equation

$$y(t) = \int_0^\infty h(\tau)\, x(t-\tau)\, d\tau \tag{1}$$

That is, the value of the output function, $y(t)$, at time t is given as a weighted linear (infinite) sum over the entire past history of the input $x(t)$. Note that Eq. (1) is a convolution integral.

If $x(t)$ is an input to the system for only a finite fixed time T, then

$$y(t) = \int_0^T h(\tau)\, x(t-\tau)\, d\tau \tag{2}$$

If $x(t)$ exists only for $t > 0$, then

$$y(t) = \int_0^t h(\tau)\, x(t-\tau)\, d\tau \tag{3}$$

The linear system may alternatively be characterized by its <u>frequency response function</u> $H(f)$ which is defined as the Fourier transform of $h(\tau)$, namely,

$$H(f) = \int_0^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{4}$$

where f is measured in cycles per unit time. The lower limit is zero instead of $-\infty$ since $h(\tau) = 0$ for $\tau < 0$. The replacement of the weighting function with the frequency response function may be made since there is a one-to-one correspondence between classes of suitably restricted functions and their Fourier transforms. That is, two different weighting functions will not give the same frequency response function. The restrictions on $h(\tau)$ are that $h(\tau)$ and its derivative $h'(\tau)$ must be piecewise continuous on every finite interval (a, b), and that $|h(\tau)|$ must be integrable on $(-\infty, \infty)$. It should be noted that the frequency response function is a special case of the <u>transfer function</u> of a system given by the Laplace transform of $h(\tau)$ in which case $e^{-j2\pi f\tau}$ in Eq. (4) is replaced by $e^{-p\tau}$ where p is a general complex variable.

The frequency response function is of great interest since it contains both amplitude magnification and phase shift information. Since $H(f)$ is complex valued, the complex exponential (polar) notation may be used. That is,

$$H(f) = |H(f)|\, e^{j\phi(f)} \tag{5}$$

where $|H(f)|$ is the absolute value of $H(f)$ and $\phi(f)$ the argument of $H(f)$. The absolute value, $|H(f)|$, measures the amplitude magnification at frequency f when the input to the system is a sinusoid of frequency f while $\phi(f)$ gives the corresponding phase shift.

Certain symmetry properties are worthwhile to note, namely,

$$H^*(f) = \int_0^\infty h(\tau)\, e^{j2\pi ft}\, d\tau = H(-f) \tag{6}$$

This relation gives

$$H^*(f) = |H(f)|\, e^{-j\phi(f)}$$

$$H(-f) = |H(-f)|\, e^{j\phi(-f)}$$

which implies

$$|H(-f)| = |H(f)| \tag{7}$$

and

$$\phi(-f) = -\phi(f) \tag{8}$$

Another important relationship for constant parameter linear systems satisfying Eq. (1) is that given the input $x(t)$, the weighting function $h(t)$, and output $y(t)$, then their Fourier transforms satisfy the simple product relation

$$Y(f) = H(f)\, X(f) \tag{9}$$

This is a general result for convolution integrals like Eq. (1). It follows that if one linear system described by $H_1(f)$ is followed in succession by another linear system $H_2(f)$, then the over-all system is described by $H(f)$ where

$$H(f) = H_1(f)\, H_2(f) \tag{10}$$

if there is no loading or feedback between the two systems. This implies

$$|H(f)| = |H_1(f)|\,|H_2(f)| \tag{11}$$

$$\phi(f) = \phi_1(f) + \phi_2(f) \tag{12}$$

so that in cascaded linear systems without loading or feedback, the amplification factors multiply and the phase shifts add.

3. RELATION TO POWER SPECTRA AND CROSS-POWER SPECTRA

Assume that $x(t)$ is a representative member from a stationary random process with zero mean value. Then the same properties are true (see Ref. [1]) for the output $y(t)$ of the linear system. A very important relation then holds between the ordinary one-sided power spectral density functions $G_x(f)$, $G_y(f)$, defined for $f \geqq 0$, and the frequency response function $H(f)$. This relation is

$$G_y(f) = |H(f)|^2 G_x(f) \tag{13}$$

That is, at any given fixed frequency f, knowledge of two of the quantities determines the third. Note, however, that the phase shift, $\phi(f)$, does not enter into this relationship.

However, one may verify (see Ref. [1]) that the complete frequency response function is related to the input spectral density $G_x(f)$ and to the cross-power spectral density $G_{xy}(f)$ (between the input and output). This simple (complex valued) formula is

$$G_{xy}(f) = H(f)\, G_x(f) \tag{14}$$

Rewriting in the complex exponential notation,

$$|G_{xy}(f)| e^{j\theta(f)} = G_x(f)\, |H(f)|\; e^{j\phi(f)}$$

which implies

$$|G_{xy}(f)| = G_x(f)\, |H(f)| \tag{15}$$

and

$$\theta(f) = \phi(f) \tag{16}$$

Therefore, with knowledge of the input power spectra and the cross-power spectra, the frequency response function is completely determined both as to amplitude magnification and phase shift. Note that several physical applications are immediately suggested. For example, the simple and cross-power spectra for random inputs may immediately be applied (if known either theoretically or experimentally) to completely determine the constant parameter linear system in terms of its frequency response function. Knowledge of the amplitude magnification and phase shift of the system may then be applied to various engineering problems such as specificiations writing for a piece of equipment to be located on this system. Also, for example, time delays occurring in a system may be determined with knowledge of the phase shift which occurs. That is, phase shift (cycles) divided by frequency (cycles/unit time) gives the time shift. One must always bear in mind the assumptions being made when applying any of the above formulas to physical problems. It must always be remembered that the above results assume

1) a constant parameter linear system,

2) an input which is a stationary random process.

4. COHERENCE FUNCTIONS

The coherence function is a real valued quantity $\gamma_{xy}^{2}(f)$, defined as

$$\gamma_{xy}^{2}(f) = \frac{|G_{xy}(f)|^{2}}{G_{x}(f)\, G_{y}(f)} \tag{17}$$

The coherence function may be thought of as the ratio of two estimates of the square of the transfer function gain factor. To be specific, consider Eq. (13) as giving one estimate

$$|\widehat{H(f)}|_{1}^{2} = \frac{G_{y}(f)}{G_{x}(f)} \tag{18}$$

and Eq. (15) as giving a second estimate

$$|\widehat{H(f)}|_{2}^{2} = \frac{G_{xy}(f)^{2}}{G_{x}^{2}(f)} \tag{19}$$

The hat ($\wedge$) above the symbols is to indicate "estimate of". If one takes the ratio of these two estimates, the coherence function is then obtained

$$\frac{|\widehat{H(f)}|_{2}^{2}}{|\widehat{H(f)}|_{1}^{2}} = \frac{|G_{xy}(f)|^{2}\, G_{x}(f)}{G_{x}^{2}(f)\, G_{y}(f)} = \gamma_{xy}^{2}(f) \tag{20}$$

One should note from these equations the absolute necessity of satisfying underlying assumptions when applying various formulas. The transfer function is an inherent property of a linear system, and although Eqs. (13) and (15) give formulas for the transfer function of a linear system under the proper conditions, they are not correct formulas under other situations. This will be specifically shown in the subsequent discussion.

The cross-power spectra may be shown to satisfy the inequality

$$|G_{xy}(f)|^2 \leq G_x(f)\, G_y(f) \tag{21}$$

which implies

$$0 \leq \gamma_{xy}^2(f) \leq 1 \tag{22}$$

Note that for a linear system, Eqs. (13) and (15) apply and may be substituted into Eq. (17) which gives

$$\gamma_{xy}^2(f) = \frac{|G_{xy}(f)|^2}{G_x(f)\, G_y(f)} \quad \frac{G_x^2(f)\,|H(f)|^2}{G_x(f)\,|H(f)|^2\, G_x(f)} = 1$$

Thus, the coherence function may be thought of as a measure of linear relationship in the sense that the function attains a theoretical maximum of unity for all f in a linear system. Hence, if the coherence function is less than unity, one possible cause is that there is not complete linear dependence between input and output. However, the reverse statement does not immediately follow. That is, the above argument does not prove that if the system is nonlinear, the coherence function necessarily is less than unity.

5. NOISE MEASUREMENTS OF FREQUENCY RESPONSE FUNCTIONS

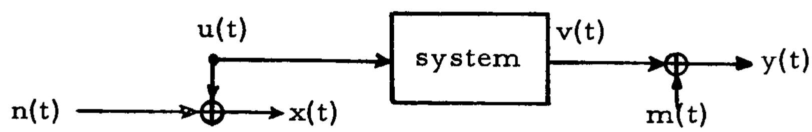
The effect of additive noise on frequency response function estimates will now be indicated. The coherence function plays a crucial role in these considerations. Three cases are considered:

1) uncorrelated noise occurring in the input measuring device,
2) uncorrelated noise occurring in the output measuring device, and
3) uncorrelated noise occurring in both the input and output measuring devices.

The third is clearly the most important and contains the other two as special cases.

5.1 NOISE IN BOTH INPUT AND OUTPUT MEASURING DEVICES

For this case the measured input $x(t)$ and measured output $y(t)$ are composed of the true signals $u(t)$ and $v(t)$ and noise components $n(t)$ and $m(t)$, respectively.



The measured input and output are given by

$$x(t) = u(t) + n(t)$$
$$y(t) = v(t) + m(t) \tag{23}$$

The spectral relations are

$$G_x(f) = G_u(f) + G_n(f)$$
$$G_y(f) = G_v(f) + G_m(f) \tag{24}$$
$$G_{xy}(f) = G_{uv}(f)$$

The coherence function for this case is

$$\gamma_{xy}^2(f) = \frac{|G_{xy}(f)|^2}{G_x(f)\,G_y(f)} = \frac{|G_{uv}(f)|^2}{\left[G_u(f) + G_n(f)\right]\left[G_v(f) + G_m(f)\right]}$$

$$= \frac{|G_{uv}(f)|^2}{G_u(f)\,G_v(f)\left[1 + \dfrac{G_u(f)\,G_m(f)}{G_u(f)\,G_v(f)} + \dfrac{G_n(f)\,G_v(f)}{G_u(f)\,G_v(f)} + \dfrac{G_n(f)\,G_m(f)}{G_u(f)\,G_v(f)}\right]}$$

$$= \frac{1}{1 + (N_1/G_1) + (N_2/G_2) + (N_1/G_1)(N_2/G_2)} < 1 \qquad (25)$$

where

$$N_1 = G_n(f) \qquad G_1 = G_u(f)$$

$$N_2 = G_m(f) \qquad G_2 = G_v(f)$$

This formula illustrates the behavior that would be expected when reasoning from the two simpler cases, namely, as the instrument noise to input and output signal ratio decreases, the coherence function approaches unity.

Simple formulas directly relating the coherence function and gain factor estimates to the true gain factor do not exist for this general case as will be shown to exist for the special cases. However, certain formulas are useful and are given below.

$$|\widehat{H(f)}|_2 = \frac{|G_{xy}(f)|}{G_x(f)} = \frac{|G_{uv}(f)|}{G_u(f) + G_n(f)} = \frac{|H(f)|}{\left[1 + G_n(f)/G_u(f)\right]}$$

or

$$|H(f)| = |\hat{H}(f)|_2 \left(1 + \left[G_n(f)/G_u(f) \right] \right) \tag{26}$$

Also,

$$\gamma_{xy}^2(f)\,|\hat{H}(f)|_1^2 = \frac{|G_{xy}(f)|^2}{G_x G_y} \; \frac{G_y}{G_x} = \frac{G_u^2(f)\,|H(f)|^2}{G_x G_y} \; \frac{G_y}{G_x}$$

$$= |H(f)|^2 \; \frac{G_u^2(f)}{G_x^2(f)}$$

or

$$|H(f)|^2 = \gamma_{xy}^2(f) \; \frac{G_x^2(f)}{G_u^2(f)} \; |\hat{H}(f)|_1^2 \tag{27}$$

5.2 NOISE IN INPUT MEASUREMENT ONLY

The special case of noise in the input measurement only corresponds to $N_2 = G_m(f) = 0$ which implies $y(t) = v(t)$. The formula for the coherence function given by Eq. (25) therefore becomes

$$\gamma_{xy}^2(f) = \frac{1}{1 + \left[G_n(f)/G_u(f) \right]} < 1 \tag{28}$$

This relationship shows clearly that any noise present in the input measuring device reduces the coherence function to less than unity. Also, as the input signal to measuring device noise ratio becomes small, the coherence function becomes small. If the noise power spectral density function is much less than the signal power spectral density function, that is $G_n(f) \ll G_u(f)$, then Eq. (28) may be put in a simpler form, namely,

$$\gamma_{xy}^2(f) \approx 1 - \left[G_n(f)/G_u(f) \right] \tag{29}$$

Equation (26) which relates $|H(f)|$ to $|\hat{H}(f)|_2$ simplifies for this case to

$$|H(f)| = |\hat{H}(f)|_2 \frac{1}{\gamma_{xy}^2(f)} \tag{30}$$

Also, Eq. (27) relating $|H(f)|$ to the first estimate, $|\hat{H}(f)|_1$, may be simplified by noting that

$$\gamma_{xy}^2 = \frac{G_u(f)}{G_u(f) + G_n(f)} = \frac{G_u(f)}{G_x(f)} \tag{31}$$

The following result is then obtained.

$$|H(f)|^2 = \gamma_{xy}^2(f) \frac{1}{\left[\gamma_{xy}^2(f)\right]^2} |\hat{H}(f)|_1^2 = \frac{1}{\gamma_{xy}^2(f)} |\hat{H}(f)|_1^2 \tag{32}$$

5.3 NOISE IN OUTPUT MEASUREMENT ONLY

The special case of noise in the output measurement only corresponds to $N_1 = G_n(f) = 0$ which implies $x(t) = u(t)$. It follows that the coherence function between $x(t)$ and $y(t)$ is given by the following special case of Eq. (25)

$$\gamma_{xy}^2(f) = \frac{G_v(f)}{G_y(f)} = \frac{1}{1 + \left[G_n(f)/G_v(f)\right]} < 1 \tag{33}$$

As in the first special case, if any noise is present, the coherence function is strictly less than one, and is inversely proportional to the output measuring device noise to true output signal ratio.

If $G_n(f) << G_v(f)$, then

$$\gamma_{xy}^2(f) \approx 1 - \left[G_n(f)/G_v(f) \right] \tag{34}$$

The specialized relations analogous to Eqs. (30) and (32) are:

$$|H(f)|^2 = |\widehat{H(f)}|_1^2 \ \gamma_{xy}^2(f) \tag{35}$$

$$|H(f)| = |\widehat{H(f)}|_2 \tag{36}$$

The above relation, Eq. (36), would appear to indicate that the cross spectral estimate gives a direct measure of the gain factor. However, it is pointed out in Section 6 that reduced statistical confidence must be placed on the measurement when the coherence function becomes less than unity.

6. CONFIDENCE LIMITS BASED ON COHERENCE FUNCTION

For the cases considered above, an estimate of the true frequency response function may be obtained from the measured functions $G_{xy}(f)$ and $G_x(f)$. Let the measured frequency response function be

$$\hat{H}(f) = \frac{G_{xy}(f)}{G_x(f)} \left|\hat{H}(f)\right| e^{j\phi(f)} \tag{37}$$

It is mentioned in Section 4.3 that although Eq. (36) apparently gives a direct estimate of H(f), reduced statistical confidence must be placed on the results. This is illustrated as follows. It has been shown (Ref. [2]) that to a very close approximation,

$$\text{Prob}\left[\left|\frac{\hat{H}(f) - H(f)}{H(f)}\right| < \sin\epsilon \quad \text{and} \quad \left|\hat{\phi}(f) - \phi(f)\right| < \epsilon\right]$$
$$\approx \left[\frac{1 - \gamma_{xy}^2(f)}{1 - \gamma_{xy}^2(f)\cos^2\epsilon}\right]^k \tag{38}$$

where k is the number of degrees of freedom (d.f.).

The number k is given by

$$k = 2BT = \frac{2N}{m} \tag{39}$$

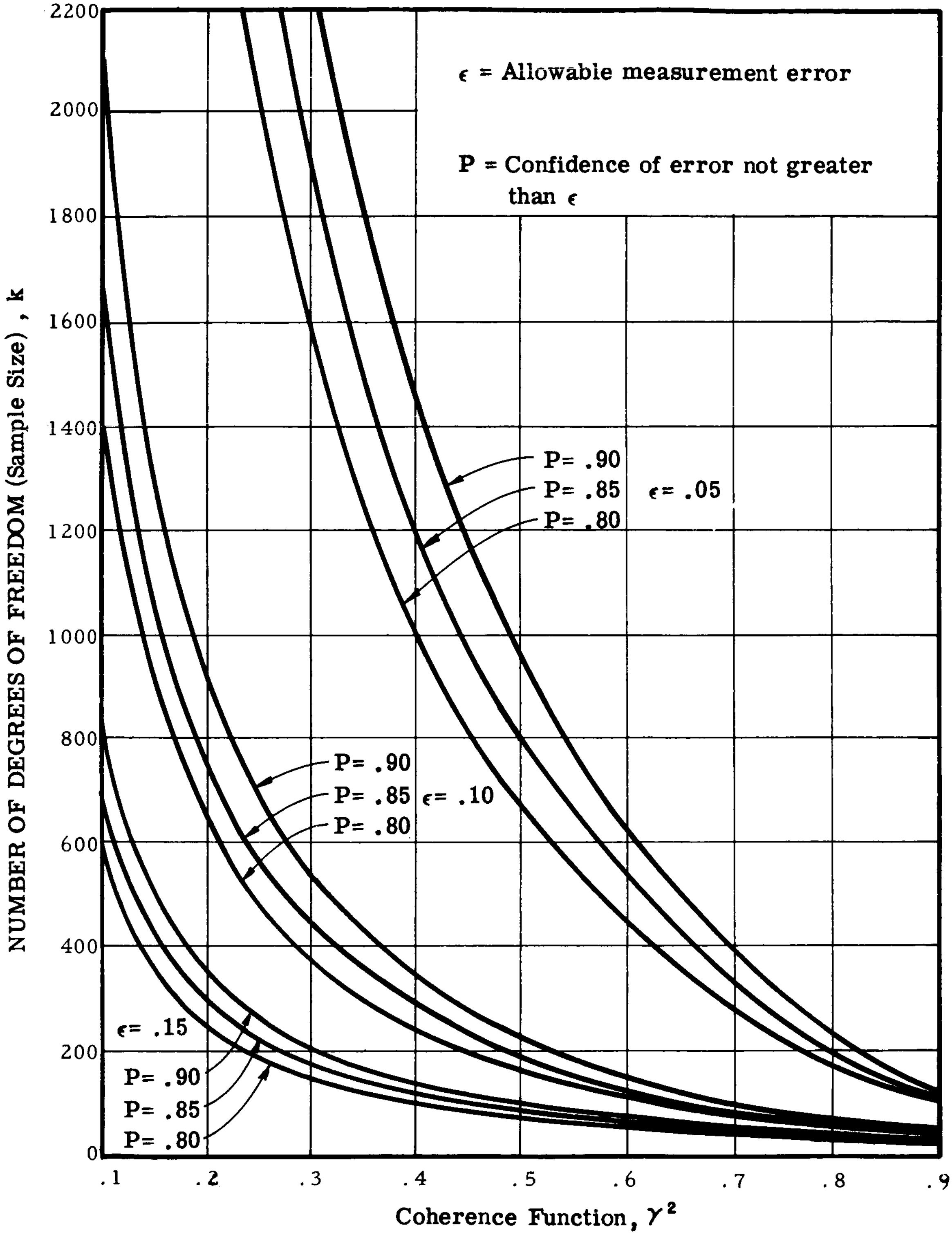
B = bandwidth

T = total record length in time

N = total number of observations

m = maximum lag number in autocorrelation estimate.

Equation (38) is displayed in Figure 1 which follows, with k as a function of $\gamma_{xy}^2(f)$. Figure 1 gives three sets of curves, one set for



DATA FOR FREQUENCY RESPONSE FUNCTION MEASUREMENT CONFIDENCE

FIGURE 1

$P = .90$, $P = .85$, and $P = .80$ when $\epsilon = .05$ radians; one set when $\epsilon = .10$ radians; and, one set when $\epsilon = .15$ radians. Since $\sin \epsilon \approx \epsilon$ for these small values of ϵ, the curves are satisfactory for a gain factor accuracy of 5, 10, and 15 percent, and a phase angle accuracy of .05, .10, and .15 radians which are approximately 2.9, 5.7, and 8.6 degrees.

The application of these curves to determine a sample size necessary to measure a frequency response function with a desired accuracy is somewhat limited at times. This is due to the fact that the coherence function is not known in advance, and therefore must be estimated. However, a conservative choice is usually in order. In this case the above relations will be practical guidelines.

6.1 APPLICATION TO ITERATIVE DETERMINATION OF H(f)

A possible application of the coherence function is as follows. Suppose a linear system is under consideration, and it is desired to estimate the frequency response function with a known accuracy. First, one would measure the coherence function by measuring the input and output power spectra separately as well as the input/output cross-power spectra. Now, with a first estimate of the coherence function, the approximate number of degrees of freedom needed to measure the frequency response function to the accuracy desired would be determined. Next, H(f) is estimated under these experimental conditions. A new coherence function measurement would now be available giving improved information and the process could be repeated, continuing the iterations until desired results were obtained.

6.2 NUMBER OF DEGREES-OF-FREEDOM FOR GIVEN CONFIDENCE

A second application of Figure 1 is as follows. Suppose the measuring instrument noise is known, or is estimated. Also, assume that based on this knowledge and approximate expected power spectra of the input and output, the coherence function of the system is estimated to be $\gamma^2 = 0.8$. Now assume that a maximum 5 percent error in the gain factor measurement with a corresponding maximum three-degree error in the phase is considered acceptable when there is a confidence of 90% of measuring these quantities that accurately. That is, $\gamma_{xy}^2(f) = .8$, $\epsilon = .05$, and $P = .90$. How many degrees of freedom are needed for the measurements? Figure 1 is entered at the $\gamma^2 = .8$ value, and the intersection with the top curve corresponding to $P = .90$ and $\epsilon = .05$ is noted. The value of k is then read off the vertical scale which is approximately $k = 240$. Therefore, about 240 d.f. are needed to measure the frequency response function under these given conditions.

6.3 ELIMINATION OF INSTRUMENT NOISE IN MEASUREMENTS OF H(f)

As can be seen from the preceding analysis, the coherence function is a useful quantity in the general consideration of frequency response functions and their measurement. If the noise in the input/output measuring equipment is known, then the frequency response function can be properly determined. However, from the formulas given, quite misleading and biased results could be obtained if no attention is paid to measurement noise. For example, assume one wants to experimentally determine in the laboratory the frequency response function of a linear system. Assume the noise in the output measuring device is known to be negligible, but the input device noise is not. Then the formulas for Case 1 would apply. First one must determine the input measuring device noise power spectral density which should be approximately constant for most situations. Then one must apply a stationary random input to the system and determine the input and output power spectral density functions. From knowledge of these quantities, Eqs. (18), (28), and (32) could be applied to determine the gain factor as well as knowledge of the uncertainty in its measurement.

7. MULTIPLE INPUT LINEAR SYSTEMS

The situation of a linear system responding to multiple inputs will now be considered. It will be assumed that N inputs are occurring with a single output being measured. It will be shown that the coherence function plays an important role for this analysis. To be specific, if it is assumed that the particular system between one of the inputs and the output is linear constant parameter, and that negligible measurement noise is present, then a low coherence function between this input and the output will serve to indicate the presence of other inputs which contribute to the output but are not being considered.

7.1 MULTIPLE CORRELATION AND POWER SPECTRA RELATIONS

Consider a constant parameter linear system with N inputs $x_i(t)$, $i = 1, 2, \ldots, N$ and one measured output $y(t)$. The assumption will be made that the output may be considered as the sum of the N outputs $y_i(t)$, $i = 1, 2, \ldots, N$. That is

$$y(t) = \sum_{i=1}^{N} y_i(t) \tag{40}$$

where $y_i(t)$ is defined as that part of the output which is produced by the ith input, $x_i(t)$, when all the other inputs are zero. See Figure 2.

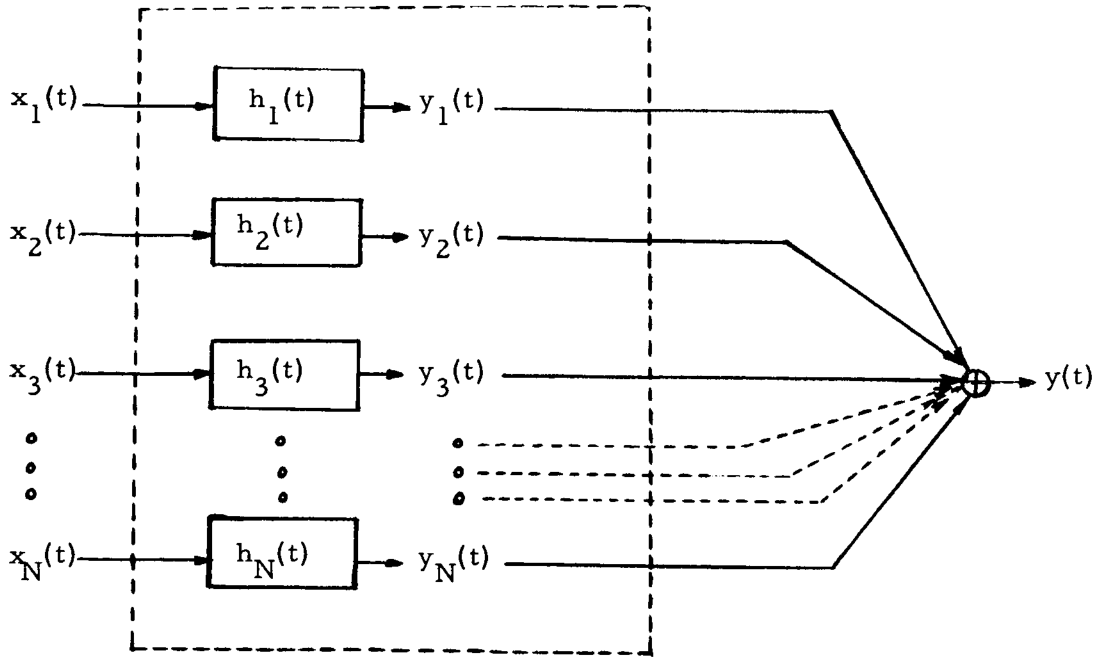


Figure 2. Block Diagram of Multiple Input Linear System

The function $h_i(\tau)$ is defined as the weighting function which is associated with the input $x_i(t)$. Hence, $y_i(t)$, from Eq. (1), is given by

$$y_i(t) = \int_0^{\infty} h_i(\tau)\, x_i(t - \tau)\, d\tau \tag{41}$$

Also, the frequency response function is given by Eq. (4) and the relation between Fourier transforms of the input and output is given by Eq. (9), namely,

$$Y_i(f) = H_i(f)\, X_i(f) \tag{42}$$

The Fourier transform, $Y(f)$, of the total output then is

$$Y(f) = \sum_{i=1}^{N} Y_i(f) = \sum_{i=1}^{N} H_i(f)\, X_i(f) \tag{43}$$

Assume next that the $x_i(t)$ are representatives from stationary random processes with mean values m_i, that is,

$$m_i = E\left[x_i(t)\right] \tag{44}$$

Recalling that the expected value operator is linear, the expected value of $y(t)$ then is obtained as

$$E\left[y(t)\right] = E\left[\sum_{i=1}^{N}\int_0^\infty h_i(\tau)\, x_i(t-\tau)\, d\tau\right]$$

$$= \sum_{i=1}^{N}\int_0^\infty h_i(\tau)\, E\left[x_i(t-\tau)\right] d\tau$$

$$= \sum_{i=1}^{N} m_i \int_0^\infty h_i(\tau)\, d\tau = \sum_{i=1}^{N} m_i H_i(0) \tag{45}$$

The autocorrelation function, $R_{yy}(\tau)$, may also be computed. Assuming that the process is stationary, one finds

$$R_{yy}(\tau) = E\left[y(t)\, y(t+\tau)\right] = E\left[\sum_i y_i(t) \sum_j y_j(t+\tau)\right]$$

$$= E\left[\sum_i \sum_j \int_0^\infty h_i(\alpha)\, x_i(t-\alpha)\, d\alpha \int_0^\infty h_j(\beta)\, x_j(t+\tau-\beta)\, d\beta\right]$$

$$= E \sum_i \sum_j \int_0^\infty \int_0^\infty h_i(\alpha)\, h_j(\beta)\, E\left[x_i(t-\alpha)\, x_j(t+\tau-\beta)\right] d\alpha\, d\beta$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \int_0^\infty \int_0^\infty h_i(\alpha)\, h_j(\beta)\, R_{ij}(\alpha-\beta+\tau)\, d\alpha\, d\beta \tag{46}$$

In Eq. (46), $R_{ij}(\tau)$ is defined as

$$R_{ij}(\tau) = R_{x_i x_j}(\tau) = E\left[x_i(t)\, x_j(t+\tau)\right] \tag{47}$$

Equation (46) is a general result for correlated inputs.

If it is assumed that all the various inputs are mutually uncorrelated, and in addition all have zero means, then

$$R_{ij}(\tau) = \begin{cases} R_{ii}(\tau) & \text{when} \quad i = j \\ 0 & \text{when} \quad i \neq j \end{cases} \tag{48}$$

In this case Eq. (46) simplifies to

$$R_{yy}(\tau) = \sum_{i=1}^{N} \int_0^\infty \int_0^\infty h_i(\alpha)\, h_i(\beta)\, R_{ii}(\alpha - \beta + \tau)\, d\alpha\, d\beta \tag{49}$$

The power spectra relations will now be computed. For stationary random processes the two-sided power spectral density function $S_{yy}(f)$ which is defined for $-\infty < f < \infty$ is given as the Fourier transform of the correlation function $R_{yy}(\tau)$. Therefore, the Fourier transform of $R_{yy}(\tau)$ as given by Eq. (46) will yield the desired power spectral density function. Thus,

$$S_{yy}(f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} R_{yy}(\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} \left[\sum_i \sum_j \int_0^\infty \int_0^\infty h_i(\alpha) h_j(\beta) R_{ij}(\alpha-\beta+\tau)\, d\alpha\, d\beta\right] d\tau \tag{50}$$

The one-sided realizable power spectral density function $G_{yy}(f) = 2S_{yy}(f)$ for $f \geq 0$, and is zero for $f < 0$.

To simplify Eq. (50), one notes that the factor $e^{-j2\pi f(\alpha-\beta)} e^{j2\pi f(\alpha-\beta)} = 1$ may be inserted, and the equation then becomes

$$S_{yy}(f) = \sum_i \sum_j \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} h_i(\alpha)\, e^{j2\pi f\alpha} h_j(\beta)\, e^{-j2\pi f\beta} R_{ij}(\alpha-\beta+\tau)\, e^{-j2\pi f(\alpha-\beta+\tau)}\, d\alpha\, d\beta\, d\tau \tag{51}$$

Now, the change of variable $t = \alpha - \beta + \tau$, $dt = d\tau$, is made and Eq. (51) may be factored to give

$$S_{yy}(f) = \sum_i \sum_j \int_0^{\infty} h_i(\alpha)\, e^{j2\pi f\alpha}\, d\alpha \int_0^{\infty} h_j(\beta)\, e^{-j2\pi f\beta}\, d\beta \int_{-\infty}^{\infty} R_{ij}(t)\, e^{-j2\pi ft}\, dt$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} H_i^*(f)\, H_j(f)\, S_{ij}(f) \tag{52}$$

In Eq. (52), $H_i^*(f)$ represents the complex conjugate of $H_i(f)$ and Eq. (6) has been applied. Also, $S_{ij}(f)$ represents the cross-power spectral density function between the inputs $x_i(t)$ and $x_j(t)$. That is, $S_{ij}(f) \equiv S_{x_i x_j}(f)$. In Eq. (52), realizable one-sided spectra $G_{yy}(f)$ and $G_{ij}(f)$ where $0 \leq f < \infty$, can replace $S_{yy}(f)$ and $S_{ij}(f)$ since the common factor of two cancels out. Equation (52) is a general result for correlated inputs.

This equation may now be specialized to the uncorrelated and zero means case where the autocorrelation function is given by Eq. (49). With these assumptions, Eq. (52) becomes

$$S_{yy}(f) = \sum_{i=1}^{N} |H_i(f)|^2\, S_{ii}(f) \tag{53}$$

where the fact has been employed that if z is a complex number then $zz^* = |z|^2$. One should note that the basic procedure used here for obtaining the above spectral relations are exactly the same as used in obtaining the single input/output relation given by Eq. (13). Observe that one may replace S(f) with G(f) in Eq. (53) since the factor of two will cancel. The relations given above in Eqs. (52) and (53) in fact represent the analogs to Eq. (13) with Eq. (53) exhibiting the most similar form.

7.2 MULTIPLE CROSS-CORRELATION AND CROSS-POWER SPECTRA RELATIONS

The cross-power spectral relation corresponding to Eq. (14) is obtained in a similar manner. Here, one wants to calculate the cross-power spectral density function $S_{x_i y}(f) = S_{iy}(f)$ of the output $y(t)$ with one of the inputs, say $x_i(t)$. The general result for correlated inputs is

$$S_{iy}(f) = \sum_{j=1}^{N} H_j(f) S_{ij}(f) \tag{54}$$

This equation is derived in a manner similar to the simple spectra formula as follows:

$$\begin{aligned} S_{iy}(f) &= \int_{-\infty}^{\infty} e^{-j2\pi f\tau} R_{iy}(\tau)\, d\tau \\ &= \sum_{j=1}^{N} \int_{-\infty}^{\infty} \left[\int_{0}^{\infty} e^{-j2\pi f\alpha} h_j(\alpha) \right] e^{-j2\pi f(\tau-\alpha)}\; R_{ij}(\tau-\alpha)\, d\alpha\, d\tau \\ &= \sum_{j=1}^{N} H_j(f) S_{ij}(f) \end{aligned}$$

where the fact has been employed that the cross-correlation function $R_{iy}(\tau)$ is given by

$$R_{iy}(\tau) = E\left[x_i(t)\, y(t+\tau)\right] = E\left[x_i(t) \sum_j \int_0^{\infty} x_j(t+\tau-\alpha)\, h_j(\alpha)\, d\alpha\right]$$

$$= \sum_j \int_0^{\infty} h_j(\alpha)\, E\left[x_i(t)\, x_j(t+\tau-\alpha)\right]\, d\alpha$$

$$= \sum_{j=1}^{N} \int_0^{\infty} h_j(\alpha)\, R_{ij}(\tau-\alpha)\, d\alpha \qquad (55)$$

If it is assumed that the inputs are independent and have zero means, then Eq. (55) reduces to

$$R_{iy}(\tau) = \int_0^{\infty} h_i(\alpha)\, R_{ii}(\tau-\alpha)\, d\alpha \qquad (56)$$

Hence, the cross spectral input/output formula given by Eq. (54) becomes

$$S_{iy}(f) = H_i(f)\, S_{ii}(f) \qquad (57)$$

This is a very interesting result since it implies that the frequency response characteristics for the structural path associated with the input, $x_i(t)$, can be measured by means of the cross-power spectra whether or not the other inputs are active if the inputs are all statistically independent. This result, in one sense, solves the problem of how to measure the individual frequency response functions for the case of uncorrelated inputs. However, the confidence in the measurement is determined by the coherence function as in Section 5.

7.3 MATRIX FORMULATION OF RESULTS

The preceding formulas can be expressed more concisely by the use of matrix notation and, in addition, some further results become more readily apparent. First define an N-dimensional input vector

$$x(t) = \left[x_1(t),\ x_2(t),\ \ldots,\ x_N(t)\right] \tag{58}$$

Also define an N-dimensional frequency response function vector

$$H(f) = \left[H_1(f),\ H_2(f),\ \ldots,\ H_N(f)\right] \tag{59}$$

Next, define an N-dimensional cross-power spectrum vector of the output $y(t)$ with the inputs $x_i(t)$,

$$S_{xy}(f) = \left[S_{1y}(f),\ S_{2y}(f),\ \ldots,\ S_{Ny}(f)\right] \tag{60}$$

where

$$S_{iy}(f) = S_{x_i y}(f), \quad i = 1, 2, \ldots, N \tag{61}$$

Finally, define the NxN matrix of cross-power spectra of all the inputs $x_i(t)$ by

$$S_{xx}(f) = \begin{bmatrix} S_{11}(f) & S_{12}(f) & \cdots & S_{1N}(f) \\ S_{21}(f) & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ S_{N1}(f) & \cdot & \cdot & S_{NN}(f) \end{bmatrix} \tag{62}$$

where

$$S_{ij}(f) \equiv S_{x_i x_j}(f) \quad , \quad i, j = 1, 2, \ldots, N \tag{63}$$

To illustrate the analogous relations to the one-dimensional case, consider Eq. (13). This may be rewritten as

$$G_y(f) = |H(f)|^2 G_x(f) = H(f)\, G_x(f)\, H^*(f) \tag{64}$$

where $H^*(f)$ indicates the complex conjugate of $H(f)$. Equation (52) may now be rewritten in matrix form as the quadratic form

$$S_{yy}(f) = H(f)\, S_{xx}(f)\, H^{*'}(f) \tag{65}$$

where $H^{*'}(f)$ denotes the complex conjugate transpose matrix. Writing out in full, Eq. (65) becomes

$$S_{yy}(f) = \left[H_1(f), H_2(f), \ldots, H_N(f)\right] \begin{bmatrix} S_{11}(f) & S_{12}(f) & \ldots & S_{1N}(f) \\ S_{21}(f) & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ S_{N1}(f) & \ldots & & S_{NN}(f) \end{bmatrix} \begin{bmatrix} H_1^*(f) \\ H_2^*(f) \\ \cdot \\ \cdot \\ \cdot \\ H_N^*(f) \end{bmatrix}$$

and the column vector on the right is the transposed complex conjugate of $H(f)$. That is

$$H^{*'}(f) = \begin{bmatrix} H_1^*(f) \\ H_2^*(f) \\ \cdot \\ \cdot \\ \cdot \\ H_N^*(f) \end{bmatrix} \tag{66}$$

Note that $S_{yy}(f)$ is still a scalar quantity but the other quantities are not. Equation (65) is also a proper representation of Eq. (53) where the off-diagonal elements of the matrix $S_{xx}(f)$ become zero for independent inputs.

In Eq. (54), the system of equations for $i = 1, 2, \ldots, N$ may be written as the matrix equation

$$S'_{xy}(f) = S_{xx}(f)\,H'(f) \tag{67}$$

This is equivalent to

$$\begin{bmatrix} S_{1y}(f) \\ S_{2y}(f) \\ \cdot \\ \cdot \\ \cdot \\ S_{Ny}(f) \end{bmatrix} = \begin{bmatrix} S_{11}(f) & S_{12}(f) \ldots & S_{1N}(f) \\ S_{21}(f) & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ S_{N1}(f) & \ldots & S_{NN}(f) \end{bmatrix} \begin{bmatrix} H_1(f) \\ H_2(f) \\ \cdot \\ \cdot \\ \cdot \\ H_N(f) \end{bmatrix}$$

where the column vectors are the transposed row vectors of $S_{xy}(f)$ and H(f). As before, Eq. (57) has its analogy when S_{xx} is a diagonal matrix due to independent inputs.

The matrix equation (67) may be solved for the transposed row vector $H'(f)$ if $S_{xy}(f)$ and $S_{xx}(f)$ have been measured or are known. This is, of course, a system of N simultaneous linear equations whose solution is

$$H'(f) = S^{-1}_{xx}(f)\,S'_{xy}(f) \tag{68}$$

where $S^{-1}_{xx}(f)$ represents the inverse matrix to $S_{xx}(f)$. Equation (68) gives each $H_i(f)$ as a function of the input/output cross-power spectra $S_{iy}(f)$ and the input/input cross-power spectra $S_{ij}(f)$, and holds whether or not the various inputs are correlated.

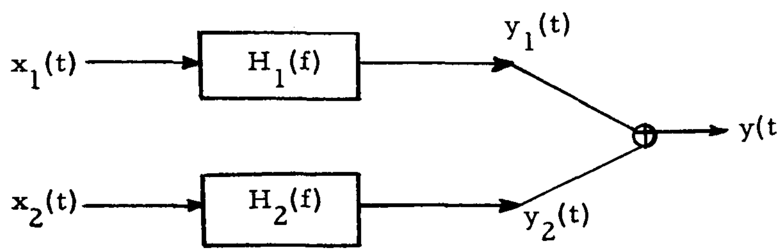
The inverse matrix, $S_{xx}^{-1}(f)$ is obtained by dividing the transposed adjoint matrix of $S_{xx}(f)$ by its determinant Δ. The adjoint matrix of $S_{xx}(f)$ is the matrix obtained by substituting the cofactor of the element $S_{ij}(f)$, written $\operatorname{cof} S_{ij}(f)$, for the element $S_{ij}(f)$. In equation form, this is

$$\begin{bmatrix} S_{11}(f) & S_{12}(f) & \cdots & S_{1N}(f) \\ S_{21}(f) & & & \\ \vdots & & & \\ S_{N1}(f) & & \cdots & S_{NN}(f) \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} \operatorname{Cof} S_{11}(f) & \operatorname{Cof} S_{21}(f) & \cdots & \operatorname{Cof} S_{N1}(f) \\ \operatorname{Cof} S_{12}(f) & & & \\ \vdots & & & \\ \operatorname{Cof} S_{1N}(f) & & \cdots & \operatorname{Cof} S_{NN}(f) \end{bmatrix} \quad (69)$$

Methods for computing the various cofactors and the determinant Δ are available in many references.

7.4 SPECIAL CASE OF TWO INPUTS

To explain the solution of Eq. (67) given by Eq. (68) for the case of two input variables, assume a two input system as pictured in the sketch below.



For this case, Eqs. (59) through (62) become

$$
\begin{aligned}
H(f) &= \left[H_1(f),\ H_2(f)\right] \\
S_{xy}(f) &= \left[S_{1y}(f),\ S_{2y}(f)\right] \\
S_{xx}(f) &= \begin{bmatrix} S_{11}(f) & S_{12}(f) \\ S_{21}(f) & S_{22}(f) \end{bmatrix}
\end{aligned}
\tag{70}
$$

The matrix Eq. (67) is

$$
\begin{bmatrix} S_{1y}(f) \\ S_{2y}(f) \end{bmatrix} = \begin{bmatrix} S_{11}(f) & S_{12}(f) \\ S_{21}(f) & S_{22}(f) \end{bmatrix} \begin{bmatrix} H_1(f) \\ H_2(f) \end{bmatrix}
\tag{71}
$$

and its solution is

$$
\begin{bmatrix} H_1(f) \\ H_2(f) \end{bmatrix} = \begin{bmatrix} S_{22}(f)/\Delta & -S_{12}(f)/\Delta \\ -S_{21}(f)/\Delta & S_{11}(f)/\Delta \end{bmatrix} \begin{bmatrix} S_{1y}(f) \\ S_{2y}(f) \end{bmatrix}
\tag{72}
$$

where Δ is the determinant of $S_{xx}(f)$ which is

$$
\Delta = S_{11}(f)\, S_{22}(f) - \left| S_{12}(f) \right|^2
\tag{73}
$$

The quantity $\left| S_{12}(f) \right|^2$ is obtained by recalling that $S_{12}(f) = S^*_{21}(f)$. Now, the quantities $H_1(f)$ and $H_2(f)$ will be solved explicitly.

The results are

$$H_1(f) = \frac{S_{22}(f)\,S_{1y}(f) - S_{12}(f)\,S_{2y}(f)}{S_{11}(f)\,S_{22}(f) - |S_{12}(f)|^2}$$

$$= \frac{S_{1y}(f)\left[1 - \dfrac{S_{12}(f)\,S_{2y}(f)}{S_{22}(f)\,S_{1y}(f)}\right]}{S_{11}(f)\left[1 - \gamma_{12}^2(f)\right]} \tag{74}$$

where

$$\gamma_{12}^2(f) = \frac{|S_{12}(f)|^2}{S_{11}(f)\,S_{22}(f)} \tag{75}$$

Also, $H_2(f)$ is given by

$$H_2(f) = \frac{S_{11}(f)\,S_{2y}(f) - S_{21}(f)\,S_{1y}(f)}{S_{11}(f)\,S_{22}(f) - |S_{12}(f)|^2}$$

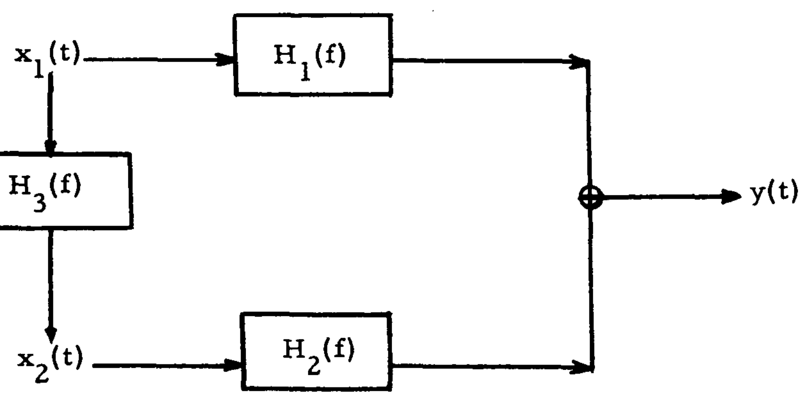
$$= \frac{S_{2y}(f)\left[1 - \dfrac{S_{21}(f)\,S_{1y}(f)}{S_{11}(f)\,S_{2y}(f)}\right]}{S_{22}(f)\left[1 - \gamma_{12}^2(f)\right]} \tag{76}$$

Note that $H_1(f)$ can be interpreted as the ratio of the cross-power spectra, $S_{1y}(f)$, to the input spectra, $S_{11}(f)$, both corrected by factors which account for the correlation between the two inputs. In the case of the denominator spectra, $S_{11}(f)$, the correction factor is seen to be the coherence function between the two inputs $x_1(t)$ and $x_2(t)$. For the special case $\gamma_{12}^2(f) = 0$, the equation reduces to the ratio of the cross-power spectra to the ordinary spectra. This is the same as the case of independent inputs since $\gamma_{12}^2(f) = 0$ implies $|S_{12}(f)|^2 = 0$ and therefore $S_{12}(f) = S_{21}(f) = 0$.

The case of $\gamma_{12}^2(f) = 1$ must be handled separately since Eq. (74) is not defined for this value of $\gamma_{12}^2(f)$. When $\gamma_{12}^2(f) = 1$,

$$S_{11}(f)\, S_{22}(f) = |S_{12}(f)|^2$$

and the determinant Δ given by Eq. (73) is therefore zero. This means that Eqs. (71) are not linearly independent and in fact one is a linear combination of the other. This is to be expected, of course, since a coherence function of unity implies complete linear dependence between $x_1(t)$ and $x_2(t)$. Therefore one could consider a linear system as existing between $x_1(t)$ and $x_2(t)$. See the sketch below.



The implication would be that the first input $x_1(t)$ was actually taking two different paths to arrive as the output $y(t)$. For this situation, a single frequency response function relates $y(t)$ to $x_1(t)$ as per Eqs. (1) and (9). The above results illustrate the interpretation of the coherence function as giving a measure of linear relation in the frequency domain.

In a general case where $\gamma_{12}^2(f) \neq 0$ or 1, the correction factor for the cross-power spectrum, $S_{1y_2}(f)$, in the numerator of Eq. (74), is seen to be of a similar form to $\gamma_{12}^2(f)$ and may be interpreted in a similar way. This point is discussed in Ref. [3] where it is shown that if one assumes a stationary Gaussian process for the inputs $x_1(t)$ and $x_2(t)$, then the numerator and denominator of Eq. (74) are the cross-power spectrum and ordinary spectrum of a "conditioned" process. That is, if one has two processes $x_1(t)$ and $x_2(t)$ which are correlated, then $x_2(t)$ may be used to "predict" $x_1(t)$ by way of a linear least squares regression equation

$$\widehat{x_1(t)} = k\, x_2(t) \tag{77}$$

where the constant k is determined from σ_{x_2}, σ_{x_1}, and the correlation coefficient $\Gamma_{x_2x_1}$ by the relation

$$k = \frac{\sigma_{x_1}}{\sigma_{x_2}} \Gamma_{x_1x_2} = \frac{\sigma_{x_1}}{\sigma_{x_2}} \left(\frac{\sigma_{x_1x_2}}{\sigma_{x_1}\sigma_{x_2}} \right) = \frac{\sigma_{x_1x_2}}{\sigma_{x_2}^2} \tag{78}$$

The result in Eqs. (77) and (78) is derived later in Eq. (83). The correlation coefficient $\Gamma_{x_2x_1}$ is defined in Eq. (81). The function $\widehat{x_1(t)}$, which is the linear least squares prediction function for $x_1(t)$, is called also the regression line for $x_1(t)$ on $x_2(t)$.

Consider a new process consisting of the difference between $x_1(t)$ and its estimate $\widehat{x_1(t)}$, that is, the residual process defined by

$$\Delta x_1(t) = x_1(t) - \widehat{x_1(t)} \tag{79}$$

Then the denominator quantity

$$S_{11\cdot 2}(f) = S_{11}(f)\left[1 - \gamma_{12}^2(f)\right] \tag{80}$$

is the spectrum of the residual process $\Delta x_1(t)$ of Eq. (79), namely, the process resulting from $x_1(t)$ after the linear least squares prediction $\hat{x}_1(t)$ has been subtracted out from $x_1(t)$. This interpretation is obtained by applying two-dimensional statistical analysis to the "spectral" variables as will be explained shortly, and is analogous to the later Eq. (84).

The "spectral" variable, X(f), where $-\infty < f < \infty$, is defined as the Fourier transform of the observed variable x(t). Note that X(f) is in general a complex number and therefore not physically observable. It may be shown that the power spectral density function $S_{xx}(f)$ is determined by the variance of the spectral variable X(f) whereas σ_x^2 is the variance of the directly observable variable x(t). Then $\gamma_{xy}^2(f)$ is determined by the square of the correlation coefficient of the observed variables x(t) and y(t).

In classical statistical analysis of real variables, the correlation coefficient between two variables x and y with zero mean values is defined as

$$\Gamma_{xy} = \frac{E\left[xy\right]}{\left(E\left[x^2\right]E\left[y^2\right]\right)^{1/2}} = \frac{\mathrm{Cov}\left[x, y\right]}{\sigma_x \sigma_y} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \tag{81}$$

where $\mathrm{Cov}\left[x, y\right] = E\left[xy\right] = \sigma_{xy}$ represents the covariance of x and y, while σ_x^2 and σ_y^2 are the variances of x and y, respectively. Note that the square of Γ_{xy} is

$$\Gamma_{xy}^2 = \frac{\left(\mathrm{Cov}\left[x, y\right]\right)^2}{\sigma_x^2 \sigma_y^2} = \frac{\sigma_{xy}^2}{\sigma_x^2 \sigma_y^2} \tag{82}$$

The notation $\sigma_x^2 = \sigma_{xx}$ and $\sigma_y^2 = \sigma_{yy}$ will be sometimes employed.

If complex numbers X, Y are being considered as sometimes occur in random process theory, the definitions above involving expected values of products must have one variable replaced by its complex conjugate. For instance, instead of xy one considers XY^* where Y^* is the complex conjugate of Y. Therefore, for complex variables, the square of the correlation coefficient becomes

$$\Gamma_{xy}^2 = \frac{E\left[XY^*\right] E\left[XY^*\right]^*}{E\left[XX^*\right] E\left[YY^*\right]} = \frac{E\left[XY^*\right] E\left[X^*Y\right]}{E\left[XX^*\right] E\left[YY^*\right]}$$

$$= \frac{\left| E\left[XY^*\right] \right|^2}{\sigma_X^2 \sigma_Y^2} = \frac{\left| \sigma_{XY} \right|^2}{\sigma_X^2 \sigma_Y^2} \tag{83}$$

Observe now that if σ_x^2 is replaced with $S_x(f)$, σ_y^2 with $S_y(f)$, and the covariance σ_{xy} with the cross-power spectral density function, $S_{xy}(f)$, then the coherence function has a form identical to that of Eq. (83). To be specific

$$\gamma_{xy}^2(f) = \frac{\left| E\left[X(f)\, Y(f)^*\right] \right|^2}{E\left[X(f)\, X(f)^*\right] E\left[Y(f)\, Y(f)^*\right]} = \frac{\left| S_{xy}(f) \right|^2}{S_x(f)\, S_y(f)} \tag{84}$$

One difference exists between Eq. (81) and correlation coefficients defined as a function of the time difference τ for a random process. For random processes, the correlation coefficient is defined as

$$\Gamma_{xy}(\tau) = \frac{R_{xy}(\tau)}{\left[R_{xx}(0)\, R_{yy}(0)\right]^{1/2}} \tag{85}$$

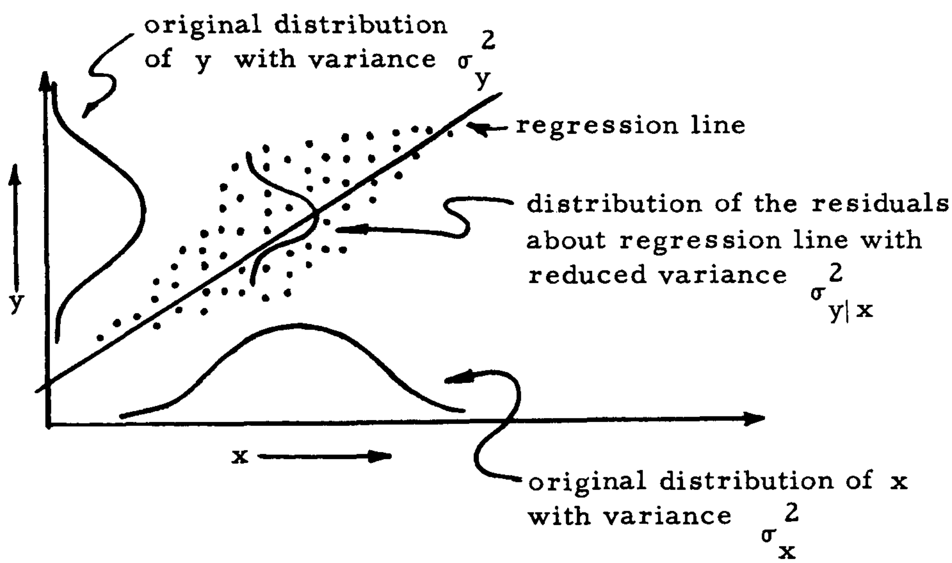
where the numerator $R_{xy}(\tau)$ is the cross-correlation function at an arbitrary τ but the denominator factors are the autocorrelation function values at $\tau = 0$ only.

If a normal distribution is assumed, the relation given by Eq. (80) is the analogy of a standard statistical result

$$\sigma_{y|x}^2 = \sigma_y^2 (1 - \Gamma_{xy}^2) \tag{86}$$

where the spectral variables replace original variables. Equation (86) says in words that the variance of the residuals about the linear least square regression line is the original variance, σ_y^2, reduced by the factor $(1 - \Gamma_{xy}^2)$.

The sketch below illustrates this idea.



To derive Eq. (86), consider the two-dimensional normal density function $p(x, y)$ defined by the equation

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \Gamma^2}} \exp\left[\frac{-1}{2(1 - \Gamma^2)}\left(\frac{x^2}{\sigma_x^2} - 2\frac{\Gamma}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right] \quad (87)$$

The mean values of the one-dimensional distributions μ_x and μ_y are assumed to be zero and Eq. (87) therefore is determined by the three remaining parameters σ_x^2, σ_y^2, and the correlation coefficient $\Gamma = \Gamma_{xy}$. One now wants to compute the variance of y given x denoted by $\sigma_{y|x}^2$. The conditional density of y given x, $p(y|x)$ is defined by

$$p(y|x) = \frac{p(x, y)}{p(x)} \quad (88)$$

where

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-x^2}{2\sigma_x^2}\right] \quad (89)$$

For the case at hand this is

$$p(y|x) = \frac{1}{\sigma_y\sqrt{2\pi}\sqrt{1 - \Gamma^2}} \exp\left[\frac{-1}{2(1 - \Gamma^2)}\left(\frac{y}{\sigma_y} - \Gamma\frac{x}{\sigma_x}\right)^2\right] \quad (90)$$

Considering this as a distribution of a single random variable y when x is a constant, the expected value and variance may be calculated as

$$E(y|x) = \int_{-\infty}^{\infty} y\, p(y|x)\, dy = \Gamma\left(\frac{\sigma_y}{\sigma_x}\right) x \quad (91)$$

and

$$\mathrm{Var}\,(y|x) = E(y^2|x) - E^2(y|x) = \sigma_{y|x}^2 = \sigma_y^2\,(1 - \Gamma^2) \quad (92)$$

The regression line for y on x (passing through the origin since the mean values μ_x and μ_y are assumed zero) is now obtained by setting y equal to the expected value of Eq. (91), namely, $\hat{y} = E(y|x)\,x$. This explains Eqs. (77) and (78). Then Eq. (92) gives the variance about this regression line which explains Eq. (80).

It is worthwhile to note that Eq. (87) can be expressed in matrix notation as

$$p(\mathbf{x}) = \frac{1}{2\pi\sqrt{|P|}} \exp\left[-\frac{1}{2}\mathbf{x}P^{-1}\mathbf{x}'\right] \tag{93}$$

where $\mathbf{x}$ is the two-dimensional vector

$$\mathbf{x} = (x, y) \tag{94}$$

and P is the covariance matrix

$$P = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{bmatrix} \tag{95}$$

with

$$\sigma_{yx} = \sigma_{xy} = E(xy) = \Gamma\sigma_x\sigma_y \tag{96}$$

The covariance matrix here is analogous to the cross-power spectra matrix defined by Eq. (62). Also the definition for an n-dimensional normal distribution is immediately obtained by Eq. (93), by just letting the vector and matrix represent n-dimensional instead of two-dinemsional quantities. The factor $(1/2\pi)$ in Eq. (93) is actually $1/(2\pi)^{n/2}$ in general.

Just as the denominator of Eq. (74) is the ordinary spectrum of the residuals, the numerator is the cross-power spectrum of the residual process. In this sense, the quantity $S_{12}(f)\,S_{2y}(f)/S_{22}(f)\,S_{1y}(f)$

is analogous to the coherence function in that it is the factor necessary to adjust the original cross spectrum $S_{1y}(f)$ to obtain the residual cross spectrum.

To be more specific in this interpretation, one must make further comparisons with classical regression analysis. However, one must consider all three variables y, x_1, and x_2. Now, one wants to calculate the covariance of the residual of y given x_2, with the residual of x_1 given x_2, when x_2 is not independent of x_1. First, note by Eqs. (77) and (78) that the residual Δx_1 is defined by

$$\Delta x_1 = x_1 - \left(\frac{\sigma_{x_1 x_2}}{\sigma_{x_2}^2}\right) x_2 \tag{97}$$

Similarly, the residual Δy is defined by

$$\Delta y = y - \left(\frac{\sigma_{y x_2}}{\sigma_{x_2}^2}\right) x_2 \tag{98}$$

Next, the covariance between Δx_1 and Δy, called the "partial covariance," is given in complex terms by

$$\sigma_{\Delta x_1 \Delta y} = \mathrm{Cov}\left[\Delta x_1, \Delta y\right] = E\left[(\Delta x_1)(\Delta y)^*\right]$$

$$= E\left[(x_1 - \frac{\sigma_{x_1 x_2}}{\sigma_{x_2 x_2}} x_2)(y^* - \frac{\sigma^*_{y x_2}}{\sigma_{x_2 x_2}} x_2^*)\right]$$

$$= E(x, y^*) - \frac{\sigma^*_{y x_2}}{\sigma_{x_2 x_2}} E(x_1 x_2^*) - \frac{\sigma_{x_1 x_2}}{\sigma_{x_2 x_2}} E(x_2 y^*) + \frac{\sigma_{x_1 x_2} \sigma^*_{y x_2}}{\sigma^2_{x_2 x_2}} E(x_2 x_2^*)$$

$$= \sigma_{x_1 y} - \frac{\sigma_{x_2 y} \sigma_{x_1 x_2}}{\sigma_{x_2 x_2}} = \sigma_{x_1 y}\left[1 - \frac{\sigma_{x_2 y} \sigma_{x_1 x_2}}{\sigma_{x_1 y} \sigma_{x_2 x_2}}\right] \tag{99}$$

For symmetry, $\sigma^2_{x_1}$ and $\sigma^2_{x_2}$ have been denoted above by $\sigma_{x_1 x_1}$ and $\sigma_{x_2 x_2}$.

The numerator of Eq. (74) is now obtained directly by replacing $\sigma_{x_1 y}$ with $S_{1y}(f)$, $\sigma_{x_1 x_2}$ with $S_{12}(f)$, $\sigma_{x_2 y}$ with $S_{2y}(f)$, and $\sigma_{x_2 x_2}$ with $S_{22}(f)$. The accepted notation for $\sigma_{\Delta x_1 \Delta y}$ is $\sigma_{x_1 y \cdot x_2}$. Therefore, one defines the numerator of Eq. (74) as the cross-spectrum of the residual input process with the residual output, namely

$$S_{1y\cdot 2}(f) = S_{1y}(f)\left[1 - \frac{S_{12}(f)\, S_{2y}(f)}{S_{22}(f)\, S_{1y}(f)}\right] \tag{100}$$

Analogous to the residual spectrum of the $x_1(t)$ variable given $x_2(t)$, one has a residual spectrum of $y(t)$ given $x_2(t)$. This is

$$S_{yy\cdot 2} = S_{yy}\left[1 - \gamma_{2y}^2\right]$$

$$= S_{yy}\left[1 - \frac{|S_{2y}|^2}{S_{yy}S_{22}}\right] \tag{101}$$

The above results and that for $S_{11\cdot 2}$ given by Eq. (80) may be obtained in a more general way by considering a matrix formulation. Note that Eq. (80) may be written in the form

$$S_{11\cdot 2}(f) = S_{11}(f)\left[1 - \gamma_{12}^2(f)\right] = S_{11}(f) - S_{12}(f)\,S_{22}^{-1}(f)\,S_{21}(f) \tag{102}$$

Consider the cross-power spectrum matrix of Eq. (62) for the three variables $y(t)$, $x_1(t)$, and $x_2(t)$ partitioned as indicated below where $x_2(t)$ is the quantity being interpreted as the predictor.

$$S_{xx} = \left[\begin{array}{cc|c} S_{yy} & S_{y1} & S_{y2} \\ S_{1y} & S_{11} & S_{12} \\ \hline S_{2y} & S_{21} & S_{22} \end{array}\right] \tag{103}$$

Now, let the 2×2 matrix in the upper left replace $S_{11}(f)$ in Eq. (100), let the 2×1 column vector in the upper right replace $S_{12}(f)$, let its complex conjugate transpose replace $S_{21}(f)$, and let $S_{22}(f)$ remain unchanged. A matrix $S_{xx\cdot 2}$ of partial cross-power spectra is then defined analogous to Eq. (102) by the following matrix equation:

$$S_{xx\cdot 2} = \begin{bmatrix} S_{yy} & S_{y1} \\ S_{1y} & S_{11} \end{bmatrix} - \begin{bmatrix} S_{y2} \\ S_{12} \end{bmatrix} S_{22}^{-1} \left[S_{y2}^{*}, S_{12}^{*} \right]$$

$$= \begin{bmatrix} S_{yy} - \dfrac{|S_{2y}|^2}{S_{22}} & S_{y1} - \dfrac{S_{y2} S_{21}}{S_{22}} \\ S_{1y} - \dfrac{S_{12} S_{2y}}{S_{22}} & S_{11} - \dfrac{|S_{12}|^2}{S_{22}} \end{bmatrix} \tag{104}$$

In the above, S instead of S(f) has been written in the interest of simplicity.

In classical statistics, one defines a squared "partial correlation coefficient" between residual variables by replacing the ordinary covariance and variances in Eq. (82) with the residual covariance and residual variances. Analogously, a partial coherence function can be defined as

$$\gamma_{1y\cdot 2}^2 = \frac{|S_{1y\cdot 2}|^2}{S_{yy\cdot 2} S_{11\cdot 2}} = \frac{\left[S_{1y} - \dfrac{S_{2y} S_{12}}{S_{22}}\right]\left[S_{y1} - \dfrac{S_{y2} S_{21}}{S_{22}}\right]}{\left[S_{yy} - \dfrac{|S_{2y}|^2}{S_{22}}\right]\left[S_{11} - \dfrac{|S_{12}|^2}{S_{22}}\right]}$$

$$= \frac{|S_{1y}|^2 \left[1 - \dfrac{S_{2y} S_{12}}{S_{1y} S_{22}}\right]\left[1 - \dfrac{S_{y2} S_{21}}{S_{y1} S_{22}}\right]}{S_{yy} S_{11} \left[1 - \gamma_{2y}^2\right]\left[1 - \gamma_{12}^2\right]} \tag{105}$$

The quantity $\gamma^2_{1y\cdot 2}(f)$ is interpreted as the coherence function between the residuals of $y(t)$ and $x_1(t)$ after the least squares prediction from $x_2(t)$ has been subtracted out. The partial correlation coefficient in classical statistics may be interpreted as a correlation coefficient between two variables when the effect of a third variable has been removed so that the spurious correlations are suppressed. The analogous interpretation may be applied to the partial coherence. That is, a high ordinary coherence between two processes could indicate a linear input/output relation between the two processes, but in reality this may be a spurious relation due to the correlation of the apparent input with another input variable. If the partial coherence function is calculated, the more realistic low coherence would be uncovered.

Alternately, the opposite effect may occur where the partial coherence will be larger than the ordinary coherence. This occurs, for example, when two separate inputs both pass through linear systems and make up the single output. Then the partial coherence between either input and the output is unity due to the linear relations, but the ordinary coherence will be less than unity. This follows because each input contributes to the output and this fact is not accounted for in the computation of the ordinary coherence function between the output and a single input. This case is illustrated by the numerical example in the next section.

Consider the special case of Eq. (105) when the two inputs x_1 and x_2 are independent, that is $S_{12}(f) = 0$. A partial coherence function $\gamma^2_{1y\cdot 2}$ is then given by

$$\gamma_{1y\cdot 2}^2 = \frac{|S_{1y}|^2}{S_{yy}S_{11}\left[1 - \frac{|S_{2y}|^2}{S_{yy}S_{22}}\right]} = \frac{\gamma_{1y}^2}{1 - \gamma_{2y}^2} \tag{106}$$

Since $\gamma_{1y\cdot 2}^2$ is strictly a coherence function it must be bounded by unity which implies

$$\gamma_{1y}^2 \leq 1 - \gamma_{2y}^2 \tag{107}$$

This is the result to be expected. Since the output $y(t)$ is made up of the sum of two outputs $y_1(t)$ and $y_2(t)$, it is natural to expect that a large coherence between one of the inputs and the over-all output necessarily implies that the contribution from the second input is small, and therefore has small coherence with the over-all output. These ideas will be illustrated with examples in Section 7.5.

For more than two inputs the nature of the solution retains the same form as in the two-input case. Also, similar basic interpretations apply through analogies with more advanced multi-variate statistical theory.

7.5 APPLICATION OF COHERENCE FUNCTIONS

7.5.1 Case of Independent Inputs

The specific example of measuring one of the N inputs relative to the output will now be discussed. Assume for this case that the N inputs $x_1(t)$, $x_2(t), \ldots, x_N(t)$ are statistically independent. Let $y(t)$ represent the output. Assume also for concreteness that it is $x_1(t)$ and $y(t)$ which are being measured. The system will be assumed to be a linear constant parameter system, and it will be assumed that the amount of noise in the input and output measuring devices is negligible. Now suppose that the physically realizable power spectra $G_{x_1 x_1}(f)$ and $G_{x_1 y}(f)$ are measured or computed from the amplitude time histories. The coherence function for this case then becomes

$$\gamma_{1y}^2(f) = \frac{|G_{1y}(f)|^2}{G_1(f)\,G_y(f)}$$

$$= \frac{|H_1(f)\,G_1(f)|^2}{G_1(f) \sum_{i=1}^{N} |H_i(f)|^2 G_i(f)}$$

$$= \frac{|H_1(f)|^2 G_1^2(f)}{|H_1(f)|^2 G_1^2(f) + |H_2(f)|^2 G_1(f)\,G_2(f) + \ldots + |H_N(f)|^2 G_1(f)\,G_N(f)}$$

$$= \frac{1}{1 + \dfrac{|H_2(f)|^2 G_2(f)}{|H_1(f)|^2 G_1(f)} + \ldots + \dfrac{|H_N(f)|^2 G_N(f)}{|H_1(f)|^2 G_1(f)}} \qquad (108)$$

Therefore, if any of the inputs other than $x_1(t)$ are nonzero, their power spectra will be nonzero and

$$\gamma_{1y}^2(f) < 1 \tag{109}$$

The amount by which the coherence function is less than unity is seen to be dependent on the size of the square of the gain factors of the frequency response functions associated with the other inputs and the size of their power spectra, relative to the power spectra and squared gain factor associated with the measured input. These factors are, of course, the output power spectra corresponding to their respective input power spectra. Therefore, the coherence function has a useful application as a tool in detecting the presence of non-measured inputs occurring in a system, along with giving some indication of their effect on the output.

7.5.2 Case of Correlated Inputs

For simplicity, the case of two correlated inputs will be considered in detail. More advanced arguments are needed to develop the general case of N correlated inputs.

Let $x_1(t)$ and $x_2(t)$ be the two inputs and $y(t) = y_1(t) + y_2(t)$ be the output as considered in Section 7.4. Assume also that only one input, say $x_1(t)$, and the output $y(t)$ are being measured. The one-sided power spectral density functions are $G_{x_1}(f) \equiv G_1(f)$, $G_{x_2}(f) \equiv G_2(f)$ for the inputs and $G_y(f)$ for the output. The frequency response functions are $H_1(f)$ and $H_2(f)$.

The coherence function between $x_1(t)$ and $y(t)$ is now computed and it will be seen to be less than unity when a second input $x_2(t)$ is active. Since $x_1(t)$ and $x_2(t)$ are assumed to be correlated, Eq. (54) is used for the cross-power spectrum and Eq. (52) applies for the output power spectrum.

The coherence function is given by

$$\gamma_{1y}^{2}(f) = \frac{|G_{1y}(f)|^2}{G_1(f)\,G_y(f)} = \frac{\left|\sum_{n=1}^{2} H_i G_{1i}\right|^2}{G_1(f) \sum_{i=1}^{2} \sum_{j=1}^{2} H_i^* H_j G_{ij}} \tag{110}$$

Writing out in full, Eq. (108) becomes

$$\gamma_{1y}^{2}(f) = \frac{\left[H_1(f)\,G_1(f) + H_2(f)G_{12}(f)\right]\left[H_1^*(f)\,G_1(f) + H_2^*(f)\,G_{12}^*(f)\right]}{G_1(f)\left[H_1^*(f)\,H_1(f)\,G_1(f) + H_1^*(f)\,H_2(f)\,G_{12}(f) + H_2^*(f)H_1(f)G_{21}(f) + H_2^*(f)H_2(f)G_2(f)\right]}$$

$$= \frac{|H_1(f)|^2 G_1^2(f) + |H_2(f)|^2 |G_{12}(f)|^2 + \left[H_1(f)G_1(f)H_2^*(f)G_{12}^*(f) + H_1^*(f)G_1(f)H_2(f)G_{12}(f)\right]}{|H_1(f)|^2 G_1^2(f) + |H_2(f)|^2 G_1(f)G_2(f) + \left[H_1^*(f)G_1(f)H_2(f)G_{12}(f) + H_1(f)G_1(f)H_2^*(f)G_{12}^*(f)\right]} \tag{111}$$

Now, the numerator of Eq. (111) is less than the denominator since the only unequal factors are $|H_2(f)|^2 |G_{12}(f)|^2$ and $|H_2(f)|^2 G_1(f)G_2(f)$. For these factors,

$$|G_{12}(f)|^2 \leq G_1(f)G_2(f) \tag{112}$$

since this is merely the statement that the coherence function between the two inputs must be less than one. Hence the coherence function $\gamma_{1y}^{2}(f)$ between the first input and the output must also be bounded by unity.

Actually, of course, any true coherence function is less than unity since it is analogous to a correlation coefficient which is bounded by unity. The explicit form of the relation given by Eq. (111)

is useful since it shows that the amount less than unity is determined to a certain extent by the magnitude of the coherence between the two inputs. That is, if the two inputs are linearly related, then the coherence function between the output and either one of the inputs is unity. However, if the coherence between the two inputs is small, then the coherence between one input and the over-all output will tend to become small. These ideas will be illustrated in the numerical example which follows.

7.6 NUMERICAL EXAMPLE FOR THE CASE OF TWO INPUTS AND ONE OUTPUT

A simplified numerical example will now be worked out to illustrate the preceding formulas and ideas. The case of two inputs and one output will be considered. Frequency response functions of low pass filters have an extremely simple analytical form, and therefore will be used for the example although no direct structural analogy exists for this case. It will be assumed that both inputs $x_1(t)$ and $x_2(t)$ are white noise and therefore have constant power spectral density functions. Two cases for the inputs will be considered:

Case 1: Uncorrelated Inputs

The two inputs will be assumed to be uncorrelated for all τ. Therefore, they have a cross-power spectral density function and coherence function identically zero for all frequencies.

Case 2: Correlated Inputs

The inputs will be assumed to have a delta function for a cross-correlation function at time $\tau_0 = 1.0$ seconds. Therefore, the cross-power spectra will be nonzero and in turn the coherence between the inputs will be nonzero.

The following values will be assumed for a numerical example.

Assumed frequency response functions:

$$H_1(f) = \frac{1}{1 + j2\pi f T_1} \quad , \qquad T_1 = .05 \text{ sec}$$
$$H_2(f) = \frac{1}{1 + j2\pi f T_2} \quad , \qquad T_2 = .10 \text{ sec} \tag{113}$$

Power spectra of inputs:

$$G_{11}(f) = k_1 = .02\, v^2/\text{cps}$$
$$G_{22}(f) = k_2 = .01\, v^2/\text{cps} \tag{114}$$

Cross-power spectrum of inputs:

Case 1: $$R_{x_1x_2}(\tau) \equiv 0, \qquad G_{12}(f) \equiv 0 \tag{115}$$

Case 2:
$$R_{x_1x_2}(\tau) = \begin{cases} k_3\delta(\tau - \tau_0) & , \quad \tau_0 = 1 \text{ sec} \\ 0 & , \quad \tau \neq \tau_0 \end{cases}$$

$$k_3 = 0.8\sqrt{k_1 k_2} \tag{116}$$

$$G_{12}(f) = k_3\, e^{-j2\pi f\tau_0}$$

$$|G_{12}(f)| = k_3 \quad ; \quad \theta(f) = -2\pi f\tau_0$$

For concreteness in the subsequent computations, it will be assumed that ten cycles per second is the frequency of interest and therefore values of the various quantities will be calculated at the point $f = 10$ cycles/second. The absolute value of the frequency response functions will be of interest in the subsequent computations and also the real and imaginary parts. These quantities are given by

$$|H_i(f)|^2 = \frac{1}{1 + (2\pi f T_i)^2} \quad , \quad i = 1, 2 \tag{117}$$

$$H_i(f) = \frac{1}{1 + (2\pi f T_i)^2} - j \frac{2\pi f T_i}{1 + (2\pi f T_i)^2} \quad , \quad i = 1, 2 \tag{118}$$

Referring to Eqs. (52) and (54) for correlated inputs, the output power spectral density function and the cross-power spectral density functions are given by

$$G_{yy} = |H_1|^2 G_{11} + |H_2|^2 G_{22} + 2 \operatorname{Re}\left[H_1 H_2 G_{12}\right] \tag{119}$$

$$G_{iy} = H_1 G_{i1} + H_2 G_{i2} \quad , \quad i = 1, 2 \tag{120}$$

Note that the fact that the sum of a complex number and its complex conjugate is equal to twice its real part has been used in obtaining Eq. (119). Evaluating Eqs. (119) and (120) for the particular example at hand, one obtains for Case 1 and Case 2 the following results:

Case 1: Uncorrelated Inputs

$$\begin{aligned} G_{yy}(f) &= \frac{k_1}{1 + (2\pi f T_1)^2} + \frac{k_2}{1 + (2\pi f T_2)^2} \\ G_{1y}(f) &= \frac{k_1}{1 + j2\pi f T_1} \quad ; \quad G_{2y}(f) = \frac{k_2}{1 + j2\pi f T_2} \end{aligned} \tag{121}$$

Case 2: Correlated Inputs

$$G_{yy}(f) = \frac{k_1}{1 + (2\pi fT_1)^2} + \frac{k_2}{1 + (2\pi fT_2)^2}$$

$$+ \frac{2k_3\left(\left[1 + (2\pi f)^2 T_1 T_2\right] \cos 2\pi f\tau_0 + 2\pi f\left[T_1 - T_2\right] \sin 2\pi f\tau_0\right)}{\left[1 + (2\pi fT_1)^2\right]\left[1 + (2\pi fT_2)^2\right]}$$

$$G_{1y}(f) = \frac{k_1}{1 + j2\pi fT_1} + \frac{k_3 e^{-j2\pi f\tau_0}}{1 + j2\pi fT_2} \tag{122}$$

$$G_{2y}(f) = \frac{k_3 e^{j2\pi f\tau_0}}{1 + j2\pi fT_1} + \frac{k_2}{1 + j2\pi fT_2}$$

Assume now that only the spectral relation given above have been measured or are known by some other means, and that the form of the frequency response function is not known. Also assume that the frequency response function is to be estimated from the first input $x_1(t)$ and the output $y(t)$ and that $x_2(t)$ is not known to exist. If one computes an estimate of the frequency response function $\widehat{H(f)}$ from the ratio of the cross-power spectrum to the input spectrum, the results of the two cases are:

Case 1: Uncorrelated Inputs.

$$\widehat{H(f)} = \frac{G_{1y}(f)}{G_{11}(f)} = \left(\frac{k_1}{1 + j2\pi fT_1}\right)\left(\frac{1}{k_1}\right) = \frac{1}{1 + j2\pi fT_1} \tag{123}$$

Case 2: Correlated Inputs

$$\hat{H}(f) = \frac{G_{1y}}{G_{11}} = \frac{1}{1 + j2\pi fT_1} + \frac{1}{k_1} \frac{k_3 e^{-j2\pi f\tau_0}}{(1 + j2\pi fT_2)} \tag{124}$$

However, the true frequency response function is given in either case by Eq. (74) which is for this example

$$H_1(f) = \frac{G_{22}(f)\, G_{1y}(f) - G_{12}(f)\, G_{2y}(f)}{G_{11}(f)\, G_{22}(f) - |G_{12}(f)|^2}$$

$$= \frac{k_2 \left[\frac{k_1}{1+j2\pi fT_1} + \frac{k_3 e^{-j2\pi f\tau_0}}{1+j2\pi fT_2}\right] - k_3 e^{-j2\pi f\tau_0} \left[\frac{k_3 e^{j2\pi f\tau_0}}{1+j2\pi fT_1} + \frac{k_2}{1+j2\pi fT_2}\right]}{k_1 k_2 - k_3^2}$$

$$= \frac{\dfrac{k_2 k_1 - k_3^2}{\left[1 + j2\pi fT_1\right]}}{k_1 k_2 - k_3^2} = \frac{1}{1 + j2\pi fT_1} \tag{125}$$

A similar result may be obtained for $H_2(f)$.

As can be seen from the above, the correct result is obtained for Case 1 when the inputs are uncorrelated, but a considerable error is introduced in Case 2 for correlated inputs.

The relative error, ϵ, for Case 2 is

$$\epsilon = \left| \frac{H_1 - \hat{H}}{H_1} \right| = \left| \frac{H_2 \dfrac{k_3 e^{-j2\pi f\tau_0}}{k_1}}{H_1} \right|$$

$$= \left| \frac{H_2}{H_1} \frac{k_3 e^{-j2\pi f\tau_0}}{k_1} \right| = \left| \frac{(1 + j2\pi fT_1)}{(1 + j2\pi fT_2)} \frac{k_3 e^{-j2\pi f\tau_0}}{k_1} \right|$$

$$= \frac{k_3 \sqrt{1 + (2\pi fT_1)^2}}{k_1 \sqrt{1 + (2\pi fT_2)^2}} \qquad (126)$$

If the above result, Eq. (126), is evaluated at f = 10 cps, the following value is obtained.

$$\epsilon = \frac{(.8)\sqrt{(.01)}}{\sqrt{.02}} \frac{\sqrt{1 + (\pi)^2}}{\sqrt{1 + (2\pi)^2}} = 0.107$$

That is, there is approximately an 11% error in the measurement of the frequency response function when the two inputs are correlated and the effect of the second input is neglected.

Consider now the computation of the ordinary coherence function between the first input and the output. The value of the coherence function for the case of the independent inputs is obtained from Eq. (108) while the coherence function for the case of correlated inputs is obtained from Eq. (110) or Eq. (111).

The results for this example are

Case 1: Uncorrelated Inputs

$$\gamma_{1y}^2(f) = \frac{1}{1 + \left(\frac{k_2}{k_1}\right)\left(\frac{1 + \left[2\pi f T_1\right]^2}{1 + \left[2\pi f T_2\right]^2}\right)} \tag{127}$$

Case 2: Correlated Inputs

$$\gamma_{1y}^2 = \frac{|H_1|^2 G_{11}^2 + |H_2|^2 |G_{12}|^2 + 2\,\mathrm{Re}\,(H_1 G_{11} H_2^* G_{12}^*)}{|H_1|^2 G_{11}^2 + |H_2|^2\, G_{11} G_{22} + 2\,\mathrm{Re}\,(H_1^* G_{11} H_2 G_{12})}$$

$$= \frac{\dfrac{k_1^2}{1 + (2\pi f T_1)^2} + \dfrac{k_3^2}{1 + (2\pi f T_2)^2} + 2\,\mathrm{Re}\,(H_1 G_{11} H_2^* G_{12}^*)}{\dfrac{k_1^2}{1 + (2\pi f T_1)^2} + \dfrac{k_1 k_2}{1 + (2\pi f T_2)^2} + 2\,\mathrm{Re}\,(H_1^* G_{11} H_2 G_{12})} \tag{128}$$

where

$$\mathrm{Re}(H_1^* G_{11} H_2 G_{12}) = \mathrm{Re}\,(H_1 G_{11} H_2^* G_{12}^*)$$

$$= \frac{k_1 k_3 \left(\left[1 + (2\pi f)^2 T_1 T_2\right] \cos 2\pi f \tau_0 + 2\pi f \left[T_1 - T_2\right] \sin 2\pi f \tau_0\right)}{\left(1 + \left[2\pi f T_1\right]^2\right)\left(1 + \left[2\pi f T_2\right]^2\right)}$$

Later parts of this example require the coherence function between the second input and the output also.

These are given by

<u>Case 1</u>: Uncorrelated Inputs

$$\gamma_{2y}^2 = \frac{1}{1 + \left(\frac{k_2}{k_1}\right)\left(\frac{1 + \left[2\pi f T_2\right]^2}{1 + \left[2\pi f T_1\right]^2}\right)} \tag{129}$$

<u>Case 2</u>: Correlated Inputs

$$\gamma_{2y}^2 = \frac{|H_2|^2 G_{22}^2 + |H_1|^2 |G_{12}|^2 + 2\,\mathrm{Re}\left[H_1 G_{21} H_2^* G_{22}\right]}{|H_2|^2 G_{22}^2 + |H_1|^2 G_{11} G_{22} + 2\,\mathrm{Re}\left[H_1^* G_{12} H_2 G_{22}\right]}$$

$$\frac{\dfrac{k_2^2}{1 + (2\pi f T_2)^2} + \dfrac{k_3^2}{1 + (2\pi f T_1)^2} + 2\,\mathrm{Re}\left[H_1 G_{21} H_2^* G_{22}\right]}{\dfrac{k_2^2}{1 + (2\pi f T_2)^2} + \dfrac{k_1 k_2}{1 + (2\pi f T_1)^2} + 2\,\mathrm{Re}\left[H_1^* G_{12} H_2 G_{22}\right]} \tag{130}$$

where

$$\mathrm{Re}\left[H_1 G_{21} H_2^* G_{22}\right] = \mathrm{Re}\left[H_1^* G_{12} H_2 G_{22}\right]$$

$$= \frac{k_2 k_3 \left(\left[1 + (2\pi f)^2 T_1 T_2\right] \cos 2\pi f \tau_0 + 2\pi f \left[T_1 - T_2\right] \sin 2\pi f \tau_0\right)}{\left(1 + \left[2\pi f T_1\right]^2\right)\left(1 + \left[2\pi f T_2\right]^2\right)}$$

These ordinary coherence functions are now evaluated at the point $f = 10$ cps. First, the following quantities will be calculated which are useful for the remainder of the computations:

$$k_1 = G_{11} = .02, \qquad k_2 = G_{22} = .01, \qquad k_3 = G_{12} = .0113137$$

$$a = |H_1|^2 = \frac{1}{1 + (2\pi f T_1)^2} = .09200$$

$$b = |H_2|^2 = \frac{1}{1 + (2\pi f T_2)^2} = .02470 \tag{131}$$

$$H_1 = a - ja\pi = .09200 - j(.289026)$$

$$H_2 = b - jb2\pi = .02470 - j(.155195)$$

From Eq. (54) one obtains for correlated inputs,

$$G_{1y} = ak_1 + bk_3 - j\left[\pi a k_1 + \pi 2 b k_3\right]$$

$$= (.02)(.09200) + (.0113137)(.02470) - j(3.14159)\left[(.02)(.09200) + 2(.0113137)(.02470)\right]$$

$$= .00211945 - j(.00753636) \tag{132}$$

$$G_{2y} = ak_3 + bk_2 - j\left[\pi a k_3 + \pi 2 b k_2\right]$$

$$= (.0113137)(.09200) + (.01)(.02470) - j(3.14159)\left[(.0113137)(.09200) + 2(.01)(.02470)\right]$$

$$= .00128786 - j(.00482190) \tag{133}$$

From Eq. (52) one obtains for correlated inputs,

$$G_{yy} = ak_1 + bk_2 + 2k_3 \,\mathrm{Re}\left[H_1^* H_2\right]$$

$$= (.02)(.09200) + (.01)(.02470) + (.0113137)(.0942558)$$

$$= .00315338 \tag{134}$$

where

$$2 \text{ Re } H_1^* H_2 = 2\Big[ab + (a\pi)(b2\pi)\Big]$$
$$= 2\Big[(.09200)(.02470) + (.289026)(.155195)\Big]$$
$$= .0942558$$

The coherence functions then are as follows:

Case 1: Uncorrelated Inputs. From Eqs. (127) and (129),

$$\gamma_{1y}^2 = \frac{1}{1 + \left(\frac{.01}{.02}\right)\left(\frac{.02470}{.09200}\right)} = .882$$

$$\gamma_{2y}^2 = \frac{1}{1 + \left(\frac{.02}{.01}\right)\left(\frac{.09200}{.02470}\right)} = .118 \qquad (135)$$

Case 2: Correlated Inputs. From Eqs. (131) to (134),

$$\gamma_{1y}^2 = \frac{|G_{1y}|^2}{G_{11}G_{yy}} = \frac{(.00211945)^2 + (.00753636)^2}{(.02)(.00315338)} = .971795$$

$$\gamma_{2y}^2 = \frac{|G_{2y}|^2}{G_{11}G_{yy}} = \frac{(.00128786)^2 + (.00482190)^2}{(.01)(.00315338)} = .789924 \qquad (136)$$

As a check on the above computations, the Case 2 coherence functions may be calculated from Eqs. (128) and (130).

$$\gamma_{1y}^2 = \frac{(.09200)(.0004) + (.02470)(.000128) + (.02)(.0113137)(.0942558)}{(.09200)(.0004) + (.02470)(.0002) + (.02)(.0113137)(.0942558)} = .971802$$

$$\gamma_{2y}^2 = \frac{(.09200)(.000128) + (.02470)(.0001) + (.01)(.0113137)(.0942558)}{(.09200)(.0002) + (.02470)(.0001) + (.01)(.0113137)(.0942558)} = .789940 \qquad (137)$$

As can be seen for Case 1, Eq. (135), the input with the greatest effect on the output has by far the largest coherence with the output. That is, the input $x_1(t)$ has the greatest power spectra and is also passing through a linear system with the largest gain factor, and is therefore contributing by far the greatest amount to the output $y(t)$. For Case 2, Eq. (136), the case of correlated inputs, there is not as large a difference as there is in the case of uncorrelated inputs. This is to be expected, of course, since there is a relatively large correlation between the two inputs to the system. One should note that, depending upon which computing procedure was used, a slightly different answer is obtained in Eqs. (136) and (137). As a matter of fact, differences are obtained in the fifth decimal place although six significant figures were maintained throughout the computation. This suggests that the formulas used to compute the quantities are not basically computationally stable, and therefore many significant figures must be maintained throughout the course of the computations in order to make sure of correct results.

The computation of the partial coherence functions for the two cases will now be illustrated. These quantities may be calculated from Eqs. (105) for Case 2, and from Eq. (106) for Case 1. One should note here that the second line of Eq. (105) is basically an unstable computational form. This is due to the factor $(1 - \gamma_{2y}^2)$ in the denominator. If the quantity γ_{2y}^2 is very close to 1, much significance will be lost when the subtraction from 1 is performed which will result in essentially dividing by zero. The righthand portion of the first line of Eq. (105) will be used for the computations in Case 2 below. The values for the partial coherence function for the two cases are given below.

Case 1: Uncorrelated Inputs. From Eq. (106),

$$\gamma^2_{1y\cdot 2} = \frac{\gamma^2_{1y}}{1 - \gamma^2_{2y}} = \frac{.882}{1 - .118} = 1.00 \tag{138}$$

Case 2: Correlated Inputs. From Eq. (105),

$$\gamma^2_{1y\cdot 2} = \frac{|G_{1y}|^2 + \dfrac{|G_{2y}|^2 |G_{12}|^2}{G^2_{22}} - 2\,\mathrm{Re}\left[\dfrac{G_{1y} G_{y2} G_{21}}{G_{22}}\right]}{\left[G_{yy} - \dfrac{|G_{2y}|^2}{G_{22}}\right]\left[G_{11} - \dfrac{|G_{12}|^2}{G_{22}}\right]}$$

$$= \frac{.0000612888 + \dfrac{(.0000249093)(.000128)}{(.0001)} - .0000884033}{(.00315338 - \dfrac{.0000249093}{.01})(.02 - \dfrac{.000128}{.01})}$$

$$= \frac{.0000047694}{.0000047696} = 1.00 \tag{139}$$

where

$$2\,\mathrm{Re}\left[\frac{G_{1y} G_{y2} G_{12}}{G_{22}}\right] = \frac{k_3}{k_1}\, 2\,\mathrm{Re}\left[G_{1y} G_{y2}\right]$$

$$= \left(\frac{.0113137}{.01}\right) 2\left[(.00211945)(.00128786) + (.00753636)(.00482190)\right]$$

$$= .0000884033$$

As expected, the values of the partial coherence function for this particular example turn out to be unity. This is due to the fact that true linear relations exist for this example and the only reason the ordinary coherence functions were less than one is due to the fact that the second input obscures the result of the first. Additional factors present in practical situations such as noise and system non-linearities, in addition to unaccounted for inputs, will also tend to maintain the value of the coherence function below unity. However, for this idealized example, calculating the partial coherence has the effect of subtracting out the effect of the second input, and therefore the true linear relation is exposed as indicated by the ideal value for the coherence function.

This particular situation cannot be expected to exist in general. As a matter of fact, the opposite effect is just as likely to occur. That is, the basic coherence function may be larger than it should be due to the fact that, although no linear relation exists between the one input being considered and the output, a second input is linearly related to the output and happens to be highly correlated with the first input. In this situation, the partial coherence function would uncover this fact by having a much smaller value than the ordinary coherence function.

It is to be observed that although a relatively simple example was chosen, the computations are still quite involved and numerous. This suggests that when the computations are being performed on experimentally obtained data where several points of the power spectral density functions are available, several specific frequencies are of interest, and possibly more than two inputs exist, that a digital computer will be an essential tool for use in the computations. This becomes apparent when one considers the fact that most equations essentially result from a system of simultaneous linear equations. This implies that the amount of computation essentially goes up as the square of the dimensionality of the problem. That is, with twice as many inputs, there will be four times as many computations.

One important aspect of the over-all problem has been neglected in the preceding discussion for the multiple input case. This is the problem of statistical measurement uncertainties. That is, one should account for the sampling distributions of the quantities being considered. The result of these considerations leads to confidence intervals about the various quantities of interest in the multiple input case, just as gain factor and phase shift confidence intervals are given in Section 6 for the single input case.

This concludes the example.

7.7 EXAMPLE OF DETECTION OF SPURIOUS COHERENCY

The preceding example in Section 7.6 will now be modified to illustrate the application of the partial coherence function in another way. In Section 7.4 it was pointed out that

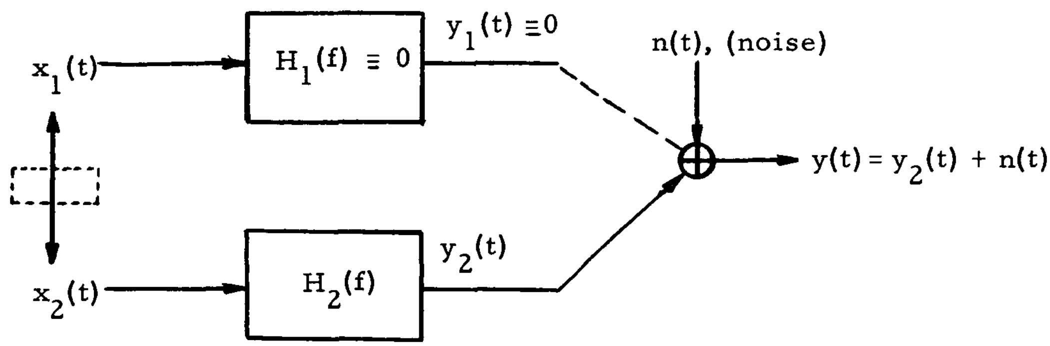
a) the partial coherence function could be greater than the ordinary coherence function, or

b) the partial coherence function could be less than the ordinary coherence function.

Case (a) was illustrated in Section 7.6, and Case (b) will now be illustrated.

The situation where a too large ordinary coherence function can occur is as follows. Suppose a signal $x_1(t)$ is being measured which is believed to be an input to a linear system, but in reality contributes nothing or very little to the output $y(t)$ which is being measured. If for some reason this signal happens to be highly correlated with a second input $x_2(t)$ which actually contributes everything to the measured output except for some measurement noise, then the signals $x_1(t)$ and $x_2(t)$ will have a large coherence which will result in a large coherence between the first input $x_1(t)$ and the output $y(t)$.

This would tend to indicate some linear relation existing between the input $x_1(t)$ and the output $y(t)$ which is in reality physically spurious. That is, the input $x_1(t)$ has no cause and effect relation with the output $y(t)$. The sketch below illustrates this situation.



As indicated in the above sketch, the first input $x_1(t)$ contributes nothing to the output. The first frequency response function $H_1(f)$ is shown to be identically zero to satisfy this situation. The second input $x_2(t)$ is assumed to be actually passing through a linear system and contributing almost everything to the output $y(t)$. The only other contribution to the output $y(t)$ is assumed to be a small amount of measurement noise denoted by $n(t)$. This noise is assumed to be statistically independent of both of the inputs and the output, and, therefore incoherent with either input or the output.

The cross-correlation between the inputs $x_1(t)$ and $x_2(t)$ is assumed to be the same as in the example of Section 7.6. Also, the power spectra of the inputs and the frequency response function $H_2(f)$ is assumed to be the same as in the previous example. These quantities are all listed below as is the assumed power spectra for the measurement noise $G_{nn}(f)$.

Assumed frequency response functions:

$$H_1(f) \equiv 0$$

$$H_2(f) = \frac{1}{1 + j2\pi f T_2} \quad , \qquad T_2 = .10 \text{ sec} \tag{140}$$

Power spectra of inputs:

$$\begin{aligned} G_{11}(f) &= k_1 = .02 \, v^2/\text{cps} \\ G_{22}(f) &= k_2 = .01 \, v^2/\text{cps} \\ G_{nn}(f) &= k_4 = .00001 \, v^2/\text{cps} \end{aligned} \tag{141}$$

Cross-power spectrum of inputs:

$$R_{x_1 x_2} = \begin{cases} k_3 \, \delta(\tau - \tau_0) & , \quad \tau_0 = 1 \text{ sec} \\ 0 & , \quad \tau \neq \tau_0 \end{cases} \tag{142}$$

$$k_3 = 0.8\sqrt{k_1 k_2}$$

$$G_{12}(f) = k_3 \, e^{-j2\pi f \tau_0}$$

From the above data, the coherence function between the two inputs is found to be

$$\gamma_{12}^2 = \frac{|G_{12}|^2}{G_{11} G_{22}} = \frac{|0.8\sqrt{k_1 k_2}|^2}{k_1 k_2} = .64 \tag{143}$$

The ordinary coherence function and the partial coherence function between the first input $x_1(t)$ and the output $y(t)$ are now of interest. The output power spectra and the cross-power spectra between the inputs and the output are needed for these calculations, and are therefore given below. These quantities may be obtained from Eqs. (119) and (120) except that the noise power spectrum must be added to the output power spectra. The noise power spectra has no effect on the cross-power spectra, however, since it is assumed to be independent of each of the inputs. The results for this example are

$$
\begin{aligned}
G_{yy} &= |H_2|^2 G_{22} + G_{nn} \\
G_{1y} &= H_2 G_{12} \\
G_{2y} &= H_2 G_{22}
\end{aligned}
\tag{144}
$$

Evaluating these quantities at the point $f = 10$ cps,

$$
\begin{aligned}
G_{yy} &= bk_2 + k_4 = (.02470)(.01) + .00001 = .0002570 \\
G_{1y} &= H_2 k_3 = (.02470 - j\,[.155195])(.0113137) = .000279448 - j(.00175583) \\
G_{2y} &= H_2 k_2 = (.02470 - j\,[.155195])(.01) = .0002470 - j(.00155195)
\end{aligned}
$$

The ordinary coherence function between the first input and the output then is

$$
\gamma_{1y}^2 = \frac{|G_{1y}|^2}{G_{11} G_{yy}} = \frac{(.00279448)^2 + (.00175583)^2}{(.02)(.002570)} = \frac{.00000316103}{.00000514000} = .614986 \tag{145}
$$

An alternative computational method for a check is as follows.

$$
\gamma_{1y}^2 = \frac{|G_{1y}|^2}{G_{11} G_{yy}} = \frac{|G_{1y}|^2}{G_{11}|H_2|^2 G_{22} + G_{11} G_{nn}} = \frac{1}{\dfrac{1}{\gamma_{12}^2} + \dfrac{G_{11} G_{nn}}{|G_{1y}|^2}} \tag{146}
$$

$$
= \frac{1}{\dfrac{1}{.64} + \dfrac{(.02)(.00001)}{.00000316103}} = \frac{1}{1.562500 + .063271} = .615093
$$

As can be seen from Eqs. (145) or (146), the value for the coherence between the first input and the output, namely 0.615, is almost the same as the coherence between the two inputs, 0.64, Eq. (143). However, the partial coherence function will now uncover the fact that this ordinary coherence function indicates a spurious relation. When Eq. (139) is applied to this example, the value of the partial coherence function, $\gamma^2_{1y\cdot 2}(f)$, is found to be zero. These calculations are given below.

$$\gamma^2_{1y\cdot 2} = \frac{|G_{1y}|^2 + \dfrac{|G_{2y}|^2|G_{12}|^2}{G_{22}^2} - 2\,\mathrm{Re}\left[\dfrac{G_{1y}G_{y2}G_{21}}{G_{22}}\right]}{\left[G_{yy} - \dfrac{|G_{2y}|^2}{G_{22}}\right]\left[G_{11} - \dfrac{|G_{12}|^2}{G_{22}}\right]}$$

$$= \frac{|H_2|^2|G_{12}|^2 + \dfrac{|H_2|^2 G_{22}^2|G_{12}|^2}{G_{22}^2} - 2\mathrm{Re}\left[\dfrac{H_2G_{12}H_2^*G_{22}G_{12}^*}{G_{22}}\right]}{\left[|H_2|^2 G_{22} + G_{nn} - \dfrac{|H_2|^2 G_{22}^2}{G_{22}}\right]\left[G_{11} - \dfrac{|G_{12}|^2}{G_{22}}\right]}$$

$$= \frac{2|H_2|^2|G_{12}|^2 - 2|H_2|^2|G_{12}{}^2}{G_{nn}\left[G_{11} - \dfrac{|G_{12}|^2}{G_{22}}\right]} = 0 \qquad (147)$$

Of course, a value of zero for the partial coherence function would not be obtained in a real situation, although a very small value might be obtained. However, this example immediately illustrates the application of the partial coherence function in uncovering this type of situation.

This concludes the example.

7.8 INFERENCES POSSIBLE FROM EXPERIMENTAL RESULTS

Consider the situation where one has measured some sort of response variables, which are stationary random processes, on three points of a structure. Assume for the present that no prior engineering information is available so as to determine which of the processes may be considered inputs or outputs associated with linear systems. The problem is therefore to determine in some way, for example, if two of the outputs may be considered as independent inputs to linear systems whose outputs add to make up a third output. Alternatively, it might be that two of the inputs are partly related and one of them passes through a linear system whose output is the third process. The application of the ideas of ordinary and partial coherence functions to this type of problem will now be illustrated.

Let the three response variables be denoted by $x_1(t)$, $x_2(t)$, and $x_3(t)$. Assume the possibility of existence of frequency response functions $H_1(f)$, $H_2(f)$, and $H_3(f)$ relating the pairs $[x_1(t), x_3(t)]$, $[x_2(t), x_3(t)]$, and $[x_1(t), x_2(t)]$ respectively. See Figure 3 below which illustrates these conjectured relations along with coherence functions between them.

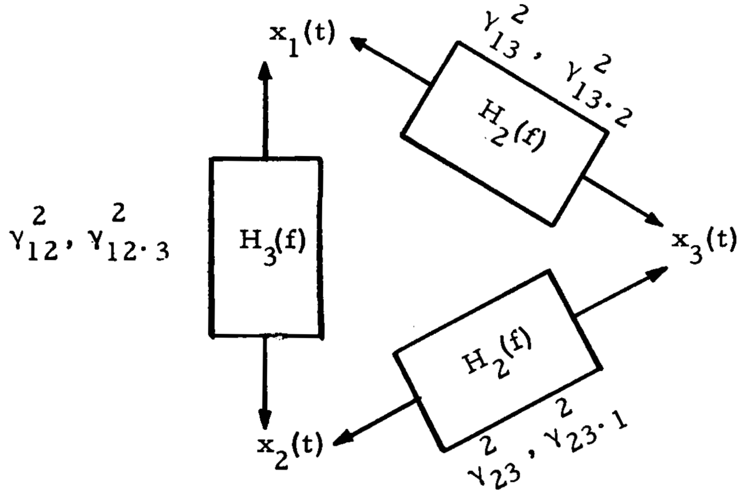


Figure 3. Possible Relationships Between Three Variables

An object of an engineering examination of these three variables $x_1(t)$, $x_2(t)$, and $x_3(t)$ might be to decide which way the arrows in the above sketch should point and whether or not one or more of the frequency response functions may be considered essentially zero. One should be aware of the fact that, mathematically speaking, it is only meaningful to calculate the degree of the linear relationships between the Fourier transformed variables $X_1(f)$, $X_2(f)$, $X_3(f)$ (i.e., spectral variables). The determination of the direction of the relation, that is, determining which is the cause and which is the effect, is of physical rather than direct mathematical significance.

Attaching a direction to the arrows in the sketch can in principle be determined from an examination of the cross-correlation function between the variables. If any significant peaks occur in the cross-correlation function, these can be defined so that positive values for the time delay, τ, will be physically meaningful. For example, if the cross-correlation function $R_{x_1x_3}(\tau)$ has a peak at $\tau_0 = 1$ sec., then the direction of the system would be from x_1 to x_3. On the other hand, if a peak of $R_{x_1 x_3}(\tau)$ occurred at $\tau_0 = -2$ sec., then the direction of travel would be from x_3 to x_1. Note, however, that this is only a physical requirement and that as far as the mathematics is concerned, there is nothing wrong with negative time delays.

Assume now that the power spectra, G_{11}, G_{22}, G_{33}, and cross-power spectra, G_{12}, G_{13}, G_{23}, have been measured. The ordinary coherence and partial coherence functions may then be calculated from Eq. (84) and Eq. (105), respectively. As a specific example, assume the results are as follows.

$$\gamma_{12}^2 = \frac{|G_{12}|^2}{G_{11}G_{22}} = 0.05 \qquad \gamma_{13}^2 = \frac{|G_{13}|^2}{G_{11}G_{33}} = 0.50 \qquad \gamma_{23}^2 = \frac{|G_{23}|^2}{G_{22}G_{33}} = 0.50$$

$$\gamma_{12\cdot 3}^2 = \frac{|G_{12\cdot 3}|^2}{G_{11\cdot 3}G_{22\cdot 3}} = 0.05 \tag{148}$$

$$\gamma_{13\cdot 2}^2 = \frac{|G_{13\cdot 2}|^2}{G_{11\cdot 2}G_{33\cdot 2}} = 0.98$$

$$\gamma_{23\cdot 1}^2 = \frac{|G_{23\cdot 1}|^2}{G_{22\cdot 1}G_{33\cdot 1}} = 0.99$$

The fact that the ordinary coherence functions γ_{13}^2 and γ_{23}^2 are somewhat less than unity while the respective partial coherences are approximately one implies that x_1 and x_2 are both linearly related to x_3. The ordinary coherence functions being less than one implies that the opposite input in each case is obscuring the true linear relation, but when these effects are subtracted out to compute the partial coherence function, the true linear relations appear. Speaking in terms of the Fourier transforms of the variables, one can visualize these relations by considering X_1 as being compared with $X_3 = X_1H_1 + X_2H_2$ for the ordinary coherence function. However, for the partial coherence function between X_1 and X_3, the effect due to X_2 is subtracted out. The comparison then is between X_1 and $X_3 - H_2X_2 = H_1X_1$ indicating the direct linear relation. From these results along with the fact that the coherence between X_1 and X_2 is essentially zero, one can reasonably infer that Figure 4 represents the physical situation. That is, x_1 and x_2 are independent inputs to linear systems whose outputs add and make up an output x_3 . Note that the values of the partial coherence functions are essential information in making these decisions. In Figure 4, the Fourier transformed variables are shown to more easily indicate the existing relations.

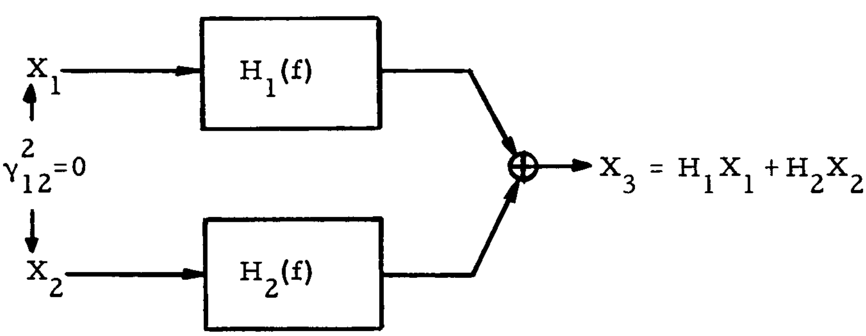


Figure 4. Possible System Diagram for Experimental Results with Independent Inputs

It is conceivable, but not likely, that the arrows in Figure 4 could point the opposite direction. For this to occur it would mean that somehow x_3 is broken up into independent components. Then these components are passed through linear systems to obtain x_1 and x_2.

As mentioned before, the cross-correlation function would give information to determine this fact. Assuming Figure 4 is accepted as being the proper system diagram, Eq. (72) could be applied to solve for the frequency response functions $H_1(f)$ and $H_2(f)$. From these, gain factors and phase shifts can be obtained.

A second possible set of experimental results might be the same as given by Eqs. (148) except that the relations between $x_1(t)$ and $x_2(t)$ are changed to

$$\gamma_{12}^2 = 0.70 \quad , \quad \gamma_{12\cdot 3}^2 = 0.70 \tag{149}$$

The interpretation could now be made that a situation similar to Figure 4 applies with certain modifications. Figure 5 indicates possibilities that might exist.

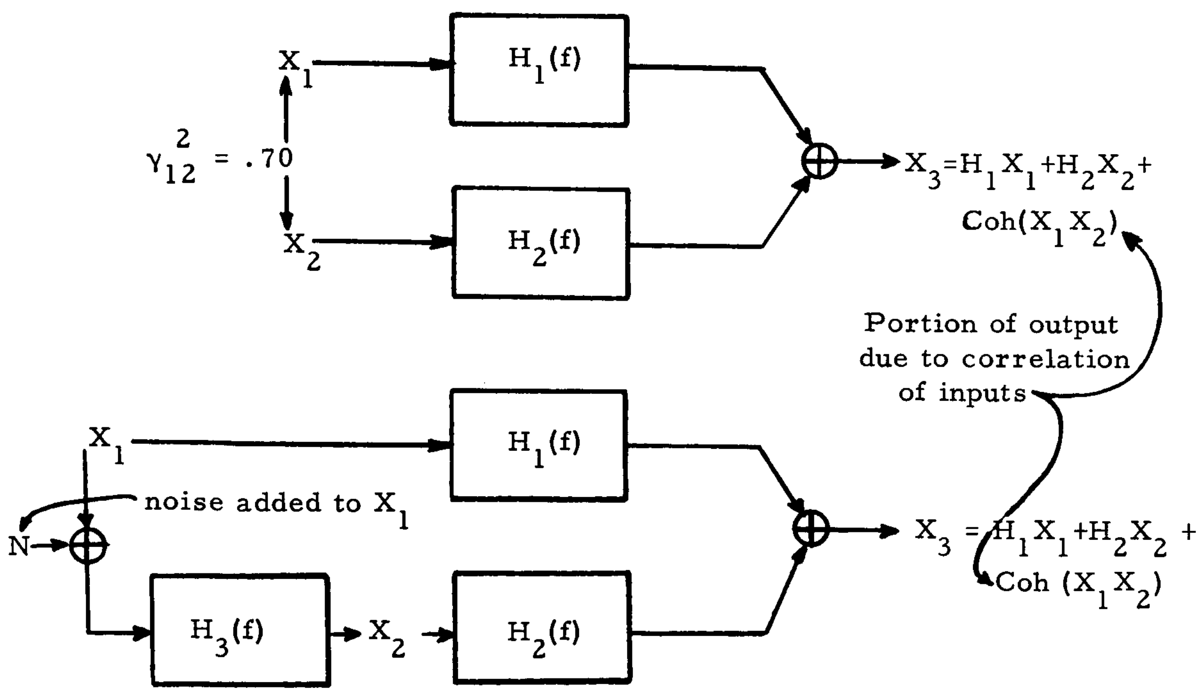


Figure 5. Linear System Example for Correlated Inputs

The coherence between the inputs could be thought of as due to a linear relation between X_1 and X_2 which is partly obscured by some independent noise. Here, the ordinary coherence function between X_1 and X_3, for example, is based on the comparison between X_1 and $X_3 = H_1X_1 + H_2X_2 + \mathrm{Coh}(X_1X_2)$. When the effect of X_2 is subtracted out, both the H_2X_2 and $\mathrm{Coh}\,(X_1X_2)$ terms drop out leaving the comparison between X_1 and H_1X_1. However, one does not necessarily need to think of some system connecting x_1 and x_2, but can consider them just as correlated inputs passing through two separate linear systems to make up x_3. Again, as for the previous example, the partial coherence functions which strongly indicate the existence of the linear systems $H_1(f)$ and $H_2(f)$ are important items of information. Note that in Figure 5, x_2 could just as well be considered an input and x_1 an output depending on the cross-correlation function. However, if these are just being thought of as correlated inputs, which is considered as an input and which is considered as an output would not make any difference.

Consider another possible set of experimentally obtained values.

$$\gamma_{12}^2 = .75 \qquad \gamma_{13}^2 = .98 \qquad \gamma_{23}^2 = .75$$

$$\gamma_{12\cdot 3}^2 = .75 \qquad \gamma_{13\cdot 2}^2 = .98 \qquad \gamma_{23\cdot 1}^2 = .02$$

The large values for γ_{13}^2 and $\gamma_{13\cdot 1}^2$ indicate a linear relation between x_1 and x_3. The value for γ_{23}^2 might indicate a linear relation between x_2 and x_3. However, the value for $\gamma_{23\cdot 2}^2$ indicates this to be spurious and due instead to the fact that x_1 and x_2 are correlated while in turn x_1 is related to x_3.

Figure 6 indicates possible causes for these data.

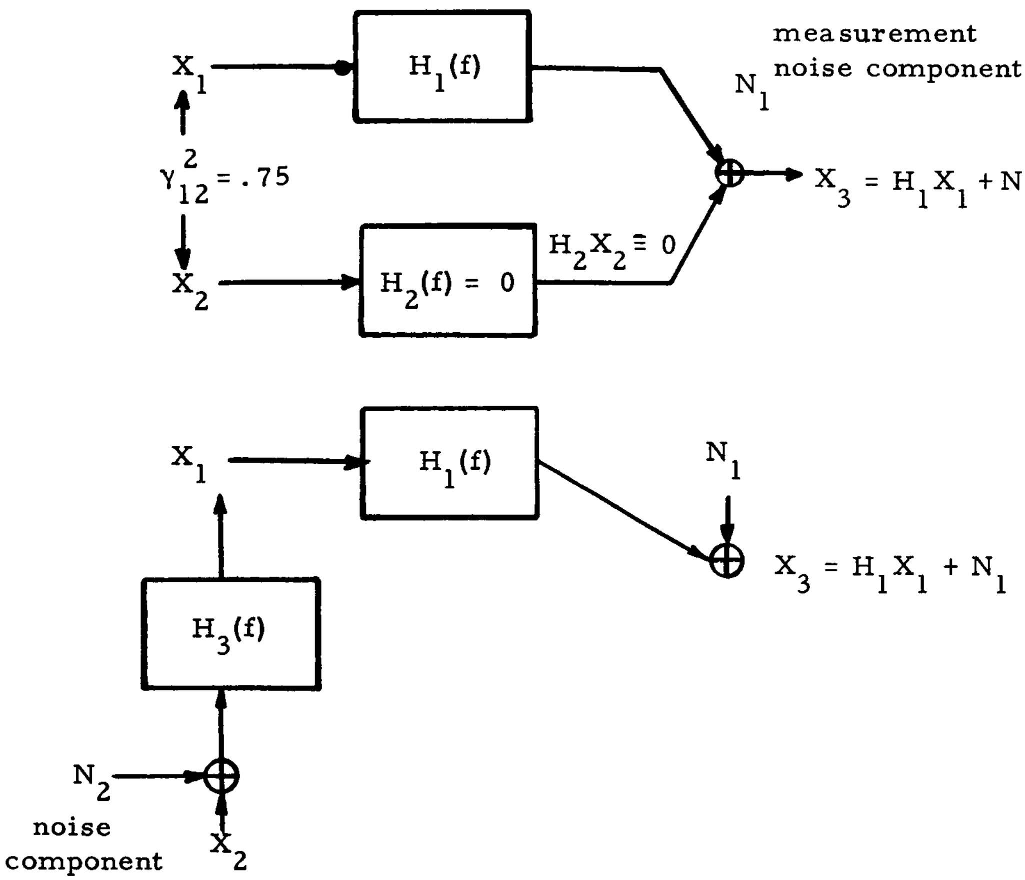


Figure 6. Linear System Example for Spurious Coherency

As before, whether or not a linear system represented by $H_3(f)$ is assumed to exist might only be a matter of personal preference.

As the number of variables increases, the possible underlying physical situations increase rapidly. It then becomes essential to have prior engineering information so as to be able to eliminate a large number of the possible variations. For example, if four variables are involved, Figure 7 indicates the possible interrelations that can exist.

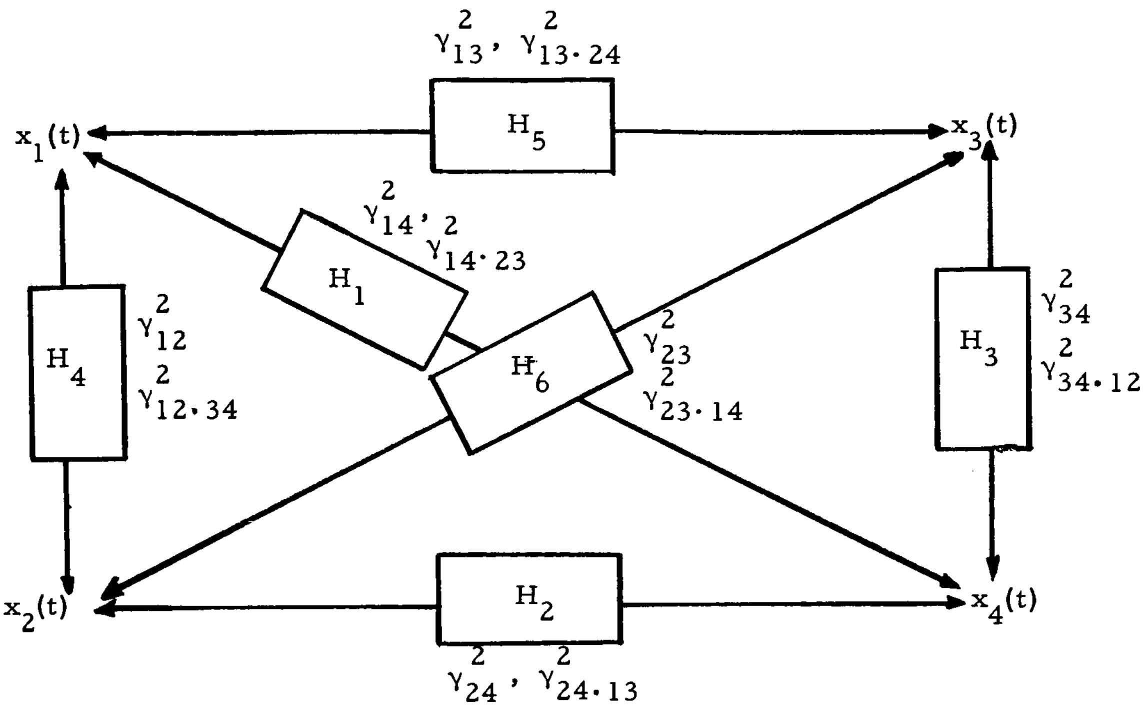


Figure 7. Possible Relationships Between Four Variables

Figure 7 shows that the number of frequency response functions has jumped from three for the case of three variables to six. A physical situation that might exist would be that x_1, x_2, and x_3 are independent inputs passing through linear systems H_1, H_2, and H_3 to make up x_4. The basic reasoning procedures which apply to the three variable case apply here. It will not be easy to eliminate possible alternatives without additional physical information. However, the cross-correlation function provides data to determine the direction of the relations, while the relative values of the ordinary and partial coherence functions provide additional valuable aids for the analysis and determination of the underlying physical conditions.

REFERENCES

1. Bendat, J.S., Principles and Applications of Random Noise Theory, John Wiley and Sons, Inc. New York. 1958.
2. Goodman, N.R., "On the Joint Estimation of the Spectra, Co-Spectrum and Quadrature Spectrum of a Two-Dimensional Stationary Gaussian Process," Scientific Paper No. 10, Engineering Statistics Laboratory, New York University. 1957.
3. Tick, L.J., "Conditional Spectra, Linear Systems, and Coherency," Chapter 13. Proceedings of the Symposium on Time Series Analysis. Edited by Murray Rosenblatt, John Wiley and Sons, Inc. New York. 1963.

NASA-Langley, 1964 CR-32